27
The Integral

> The two questions, the first that of finding the description of the curve from its elements, the second that of finding the figure from the given differences, both reduce to the same thing. From this it can be taken that the whole of the theory of the inverse method of the tangents is reducible to quadratures. (Leibniz 1673)
>
> Utile erit scribit $\int$ pro omnia. (Leibniz, October 29 1675)

27.1 Primitive Functions and Integrals

In this chapter, we begin the study of the subject of *differential equations*, which is one of the common ties binding together all areas of science and engineering, and it would be hard to overstate its importance. We have been preparing for this chapter for a long time, starting from the beginning with Chapter *A very short course in Calculus*, through all of the chapters on functions, sequences, limits, real numbers, derivatives and basic differential equation models. So we hope the gentle reader is both excited and ready to embark on this new exploration.

We begin our study with the simplest kind of differential equation, which is of fundamental importance:

> Given the **function** $f : I \to \mathbb{R}$ defined on the interval $I = [a, b]$, find a **function** $u(x)$ on I, such that the derivative $u'(x)$ of $u(x)$ is equal to $f(x)$ for $x \in I$.

We can formulate this problem more concisely as: given $f : I \to \mathbb{R}$ find $u : I \to \mathbb{R}$ such that

$$u'(x) = f(x) \tag{27.1}$$

for all $x \in I$. We call the solution $u(x)$ of the differential equation $u'(x) = f(x)$ for $x \in I$, a *primitive function* of $f(x)$, or an *integral* of $f(x)$. Sometimes the term *antiderivative* is also used.

To understand what we mean by "solving" (27.1), we consider two simple examples. If $f(x)$=1 for $x \in \mathbb{R}$, then $u(x) = x$ is a solution of $u'(x) = f(x)$ for $x \in \mathbb{R}$, since $Dx = 1$ for all $x \in \mathbb{R}$. Likewise if $f(x) = x$, then $u(x) = x^2/2$ is a solution of $u'(x) = f(x)$ for $x \in \mathbb{R}$, since $Dx^2/2 = x$ for $x \in \mathbb{R}$. Thus the function x is a primitive function of the constant function 1, and $x^2/2$ is a primitive function of the function x.

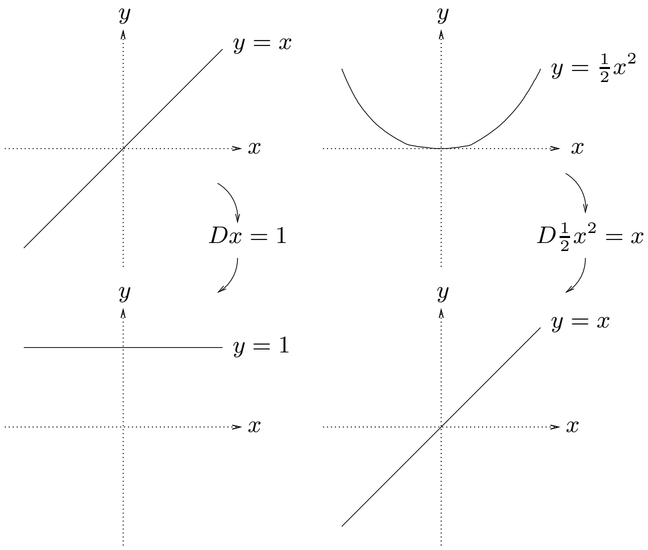

Fig. 27.1. $Dx = 1$ and $D(x^2/2) = x$

We emphasize that the solution of (27.1) is a **function** defined on an interval. We can interpret the problem in physical terms if we suppose that $u(x)$ represents some accumulated quantity like a sum of money in a bank, or an amount of rain, or the height of a tree, while x represents some changing quantity like time. Then solving (27.1) amounts to computing the total accumulated quantity $u(x)$ from knowledge of the rate of growth $u'(x) = f(x)$ at each instant x. This interpretation suggests that finding the total accumulated quantity $u(x)$ amounts to adding little pieces of momentary increments or changes of the quantity $u(x)$. Thus we expect that finding the integral $u(x)$ of a function $f(x)$ satisfying $u'(x) = f(x)$ will amount to some kind of *summation*.

A familiar example of this problem occurs when $f(x)$ is a velocity and x represents time so that the solution $u(x)$ of $u'(x) = f(x)$, represents the distance traveled by a body moving with instantaneous velocity $u'(x) = f(x)$. As the examples above show, we can solve this problem in simple cases, for example when the velocity $f(x)$ is equal to a constant v for all x and therefore the distance traveled during a time x is $u(x) = vx$. If we travel with constant velocity 4 miles/hour for two hours, then the distance traveled is 8 miles. We reach these 8 miles by accumulating distance foot-by-foot, which would be very apparent if we are walking!

An important observation is that the differential equation (27.1) alone is not sufficient information to determine the solution $u(x)$. Consider the interpretation when f represents velocity and u distance traveled by a body. If we want to know the position of the body, we need to know only the distance traveled but also the starting position. In general, a solution $u(x)$ to (27.1) is determined only up to a constant, because the derivative of a constant is zero. If $u'(x) = f(x)$, then also $(u(x) + c)' = f(x)$ for any constant c. For example, both $u(x) = x^2$ and $u(x) = x^2 + 1$ satisfy $u'(x) = 2x$. Graphically, we can see that there are many "parallel" functions that have the same slope at every point. The constant may be specified by specifying the value of the function $u(x)$ at some point. For example, the solution of $u'(x) = x$ is $u(x) = x^2 + c$ with c a constant, and specifying $u(0) = 1$ gives that $c = 1$.

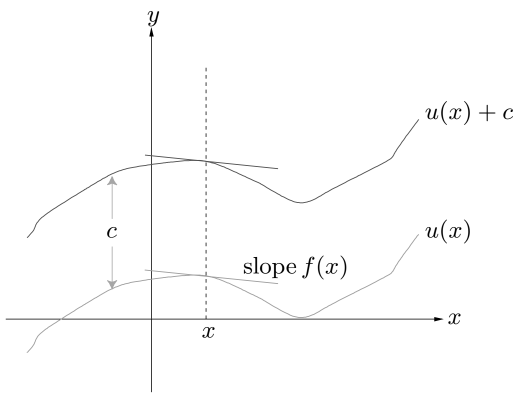

Fig. 27.2. Two functions that have the same slope at every point

More generally, we now formulate our basic problem as follows: Given $f : [a, b] \to \mathbb{R}$ and u_a, find $u : [a, b] \to \mathbb{R}$ such that

$$\begin{cases} u'(x) = f(x) & \text{for } a < x \leq b, \\ u(a) = u_a, \end{cases} \tag{27.2}$$

where u_a is a given *initial value*. The problem (27.2) is the simplest example of an *initial value problem* involving a differential equation and an initial value. The terminology naturally couples to situations in which x represents time and $u(a) = u_a$ amounts to specifying $u(x)$ at the initial time $x = a$. Note that we often keep the initial value terminology even if x represents a quantity different from time, and in case x represents a space coordinate we may alternatively refer to (27.2) as a *boundary value problem* with now $u(a) = u_a$ representing a given *boundary value*.

We shall now prove that the initial value problem (27.2) has a unique solution $u(x)$ if the given function $f(x)$ is Lipschitz continuous on $[a, b]$. This is the *Fundamental Theorem of Calculus*, which stated in words says that a Lipschitz continuous function has a (unique) primitive function. Leibniz referred to the Fundamental Theorem as the "inverse method of tangents" because he thought of the problem as trying to find a curve $y = u(x)$ given the slope $u'(x)$ of its tangent at every point x.

We shall give a constructive proof of the Fundamental Theorem, which not only proves that $u : I \to \mathbb{R}$ exists, but also gives a way to compute $u(x)$ for any given $x \in [a, b]$ to any desired accuracy by computing a sum involving values of $f(x)$. Thus the version of the Fundamental Theorem we prove contains two results: (i) the existence of a primitive function and (ii) a way to compute a primitive function. Of course, (i) is really a consequence of (ii) since if we know how to compute a primitive function, we also know that it exists. These results are analogous to defining $\sqrt{2}$ by constructing a Cauchy sequence of approximate solutions of the equation $x^2 = 2$ by the Bisection algorithm. In the proof of the Fundamental Theorem we shall also construct a Cauchy sequence of approximate solutions of the differential equation (27.2) and show that the limit of the sequence is an exact solution of (27.2).

We shall express the solution $u(x)$ of (27.2) given by the Fundamental Theorem in terms of the data $f(x)$ and u_a as follows:

$$u(x) = \int_a^x f(y)\, dy + u_a \quad \text{for } a \leq x \leq b, \tag{27.3}$$

where we refer to

$$\int_a^x f(y)\, dy$$

as the *integral* of f over the interval $[a, x]$, a and x as the *lower and upper limits of integration* respectively, $f(y)$ as the *integrand* and y the *integration variable*. This notation was introduced on October 29 1675 by Leibniz, who thought of the integral sign $\int$ as representing "summation" and dy as the "increment" in the variable y. The notation of Leibniz is part of the big success of Calculus in science and education, and (like a good cover of a record) it gives a direct visual expression of the mathematical content of the integral in very suggestive form that indicates both the construction of the integral and how to operate with integrals. Leibniz choice of notation plays an important role in making Calculus into a "machine" which "works by itself".

We recapitulate: There are two basic problems in Calculus. The first problem is to determine the derivative $u'(x)$ of a given function $u(x)$. We have met this problem above and we know a set of rules that we can use to attack this problem. The other problem is to find a function $u(x)$ given its derivative $u'(x)$. In the first problem we assume knowledge of $u(x)$ and we want to find $u'(x)$. In the second problem we assume knowledge of $u'(x)$ and we want to find $u(x)$.

As an interesting aside, the proof of the Fundamental Theorem also shows that the integral of a function over an interval may be interpreted as the area underneath the graph of the function over the interval. This couples the problem of finding a primitive function, or computing an integral, to that of computing an area, that is to *quadrature*. We expand on this geometric interpretation below.

Note that in (27.2), we require the differential equation $u'(x) = f(x)$ to be satisfied for x in the half-open interval $(a, b]$ excluding the left end-point $x = a$, where the differential equation is replaced by the specification $u(a) = u_a$. The proper motivation for this will become clear as we develop the proof of the Fundamental Theorem. Of course, the derivative $u'(b)$ at the right end-point $x = b$, is taken to be the left-hand derivative of u. By continuity, we will in fact have also $u'(a) = f(a)$, with $u'(a)$ the right-hand derivative.

27.2 Primitive Function of $f(x) = x^m$ for $m = 0, 1, 2, \dots$

For some special functions $f(x)$, we can immediately find primitive functions $u(x)$ satisfying $u'(x) = f(x)$ for x in some interval. For example, if $f(x) = 1$, then $u(x) = x + c$, with c a constant, for $x \in \mathbb{R}$. Further, if $f(x) = x$, then $u(x) = x^2/2 + c$ for $x \in \mathbb{R}$. More generally, if $f(x) = x^m$, where $m = 0, 1, 2, 3, \dots$ TSh, then $u(x) = x^{m+1}/(m+1) + c$. Using the notation (27.3) for $x \in \mathbb{R}$ we write

$$\int_0^x 1\,dy = x, \quad \int_0^x y\,dy = \frac{x^2}{2}, \tag{27.4}$$

and more generally for $m = 0, 1, 2, \dots,$

$$\int_0^x y^m\,dy = \frac{x^{m+1}}{m+1}, \tag{27.5}$$

because both right and left hand sides vanish for $x = 0$.

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27.3 Primitive Function of $f(x) = x^m$ for $m = -2, -3, \dots$

We recall that if $v(x) = x^{-n}$, where $n = 1, 2, 3, \dots,$ then $v'(x) = -nx^{-(n+1)}$, where now $x \neq 0$. Thus a primitive function of $f(x) = x^m$ for $m = -2, -3, \dots$ is given by $u(x) = x^{m+1}/(m+1)$ for $x > 0$. We can state this fact as follows: For $m = -2, -3, \dots,$

$$\int_1^x y^m \, dy = \frac{x^{m+1}}{m+1} - \frac{1}{m+1} \quad \text{for } x > 1, \tag{27.6}$$

where we start the integration arbitrarily at $x = 1$. The starting point really does not matter as long as we avoid 0. We have to avoid 0 because the function x^m with $m = -2, -3, \dots,$ tends to infinity as x tends to zero. To compensate for starting at $x = 1$, we subtract the corresponding value of $x^{m+1}/(m+1)$ at $x = 1$ from the right hand side. We can write analogous formulas for $0 < x < 1$ and $x < 0$.

Summing up, we see that the polynomials x^m with $m = 0, 1, 2, \dots,$ have the primitive functions $x^{m+1}/(m+1)$, which again are polynomials. Further, the rational functions x^m for $m = -2, -3, \dots,$ have the primitive functions $x^{m+1}/(m+1)$, which again are rational functions.

27.4 Primitive Function of $f(x) = x^r$ for $r \neq -1$

Given our success so far, it would be easy to get overconfident. But we encounter a serious difficulty even with these early examples. Extending the previous arguments to rational powers of x, since $Dx^s = sx^{s-1}$ for $s \neq 0$ and $x > 0$, we have for $r = s - 1 \neq -1$,

$$\int_1^x y^r \, dy = \frac{x^{r+1}}{r+1} - \frac{1}{r+1} \quad \text{for } x > 1. \tag{27.7}$$

This formula breaks down for $r = -1$ and therefore we do not know a primitive function of $f(x) = x^r$ with $r = -1$ and moreover we don't even know that one exists. In fact, it turns out that most of the time we cannot solve the differential equation (27.2) in the sense of writing out $u(x)$ in terms of known functions. Being able to integrate simple rational functions is special. The Fundamental Theorem of Calculus will give us a way past this difficulty by providing the means to approximate the unknown solution to any desired accuracy.

27.5 A Quick Overview of the Progress So Far

Any function obtained by linear combinations, products, quotients and compositions of functions of the form x^r with rational power $r \neq 0$ and $x > 0$, can be differentiated analytically. If $u(x)$ is such a function, we thus obtain an analytical formula for $u'(x)$. If we now choose $f(x) = u'(x)$, then of course $u(x)$ satisfies the differential equation $u'(x) = f(x)$, so that we can write recalling Leibniz notation:

$$u(x) = \int_0^x f(y)\,dy + u(0) \quad \text{for } x \geq 0,$$

which apparently states that the function $u(x)$ is a primitive function of its derivative $f(x) = u'(x)$ (assuming that $u(x)$ is defined for all $x \geq 0$ so that no denominator vanishes for $x \geq 0$).

We give an example: Since $D(1+x^3)^{\frac{1}{3}} = (1+x^3)^{-\frac{2}{3}}x^2$ for $x \in \mathbb{R}$, we can write

$$(1+x^3)^{\frac{1}{3}} = \int_0^x \frac{y^2}{(1+y^3)^{\frac{2}{3}}}\,dy + 1 \quad \text{for } x \in \mathbb{R}.$$

In other words, we know primitive functions $u(x)$ satisfying the differential equation $u'(x) = f(x)$ for $x \in I$, for any function $f(x)$, which itself is a derivative of some function $v(x)$ so that $f(x) = v'(x)$ for $x \in I$. The relation between $u(x)$ and $v(x)$ is then

$$u(x) = v(x) + c \quad \text{for } x \in I,$$

for some constant c.

On the other hand, if $f(x)$ is an arbitrary function of another from, then we may not be able to produce an analytical formula for the corresponding primitive function $u(x)$ very easily or not at all. The Fundamental Theorem now tells us how to compute a primitive function of an arbitrary Lipschitz continuous function $f(x)$. We shall see that in particular, the function $f(x) = x^{-1}$ has a primitive function for $x > 0$ which is the famous *logarithm function* $\log(x)$. The Fundamental Theorem therefore gives in particular a constructive procedure for computing $\log(x)$ for $x > 0$.

27.6 A "Very Quick Proof" of the Fundamental Theorem

We shall now enter into the proof of the Fundamental Theorem. It is a good idea at this point to review the Chapter *A very short course in Calculus*. We shall give a sequence of successively more complete versions of the proof of the Fundamental Theorem with increasing precision and generality in each step.

The problem we are setting out to solve has the following form: given a function $f(x)$, find a function $u(x)$ such that $u'(x) = f(x)$ for all x in an interval. In this problem, we start with $f(x)$ and seek a function $u(x)$ such that $u'(x) = f(x)$. However in the early "quick" versions of the proofs, it will appear that we have turned the problem around by starting with a given function $u(x)$, differentiating u to get $f(x) = u'(x)$, and then recovering $u(x)$ as a primitive function of $f(x) = u'(x)$. This naturally appears to be quite meaningless circular reasoning, and some Calculus books completely fall into this trap. But we are doing this to make some points clear. In the final proof, we will in fact start with $f(x)$ and construct a function $u(x)$ that satisfies $u'(x) = f(x)$ as desired!

Let now $u(x)$ be differentiable on $[a,b]$, let $x \in [a,b]$, and let $a = y_0 < y_1 < \ldots < y_m = x$ be a *subdivision* of $[a,x]$ into subintervals $[a,y_1)$, $[y_1,y_2),\ldots,[y_{m-1},x)$. By repeatedly subtracting and adding $u(y_j)$, we obtain the following identity which we refer to as a *telescoping sum* with the terms cancelling two by two:

$$\begin{aligned} u(x) - u(a) &= u(y_m) - u(y_0) \\ &= u(y_m) - u(y_{m-1}) + u(y_{m-1}) - u(y_{m-2}) + u(y_{m-2}) \\ &\qquad - \cdots + u(y_2) - u(y_1) + u(y_1) - u(y_0). \end{aligned} \tag{27.8}$$

We can write this identity in the form

$$u(x) - u(a) = \sum_{i=1}^{m} \frac{u(y_i) - u(y_{i-1})}{y_i - y_{i-1}} (y_i - y_{i-1}), \tag{27.9}$$

or as

$$u(x) - u(a) = \sum_{i=1}^{m} f(y_{i-1})(y_i - y_{i-1}), \tag{27.10}$$

if we set

$$f(y_{i-1}) = \frac{u(y_i) - u(y_{i-1})}{y_i - y_{i-1}} \quad \text{for } i = 1,\ldots,m. \tag{27.11}$$

Recalling the interpretation of the derivative as the ratio of the change in a function to a change in its input, we obtain our first version of the Fundamental Theorem as the following analog of (27.10) and (27.11):

$$u(x) - u(a) = \int_a^x f(y)\,dy \quad \text{where} \quad f(y) = u'(y) \quad \text{for } a < y < x.$$

In the integral notation, the sum $\sum$ corresponds to the integral sign $\int$, the increments $y_i - y_{i-1}$ correspond to dy, the y_{i-1} to the integration variable y, and the difference quotient $\frac{u(y_i)-u(y_{i-1})}{y_i - y_{i-1}}$ corresponds to the derivative $u'(y_{i-1})$.

This is the way that Leibniz was first led to the Fundamental Theorem at the age of 20 (without having studied any Calculus at all) as presented in his *Art of Combinations* from 1666.

Note that (27.8) expresses the idea that “the whole is equal to the sum of the parts” with “the whole” being equal to $u(x)-u(a)$ and the “parts” being the differences $(u(y_m)-u(y_{m-1}))$, $(u(y_{m-1})-u(y_{m-2}))$,. . . ,$(u(y_2)-u(y_1))$ and $(u(y_1)-u(y_0))$. Compare to the discussion in Chapter *A very short Calculus course* including Leibniz’ teen-age dream.

27.7 A “Quick Proof” of the Fundamental Theorem

We now present a more precise version of the above “proof”. To exercise flexibility in the notation, which is a useful ability, we change notation slightly. Let $u(x)$ be uniformly differentiable on $[a, b]$, let $\bar{x} \in [a, b]$, and let $a = x_0 < x_1 < \ldots < x_m = \bar{x}$ be a partition of $[a, \bar{x}]$. We thus change from y to x and from x to $\bar{x}$. With this notation x serves the role of a variable and $\bar{x}$ is a particular value of x. We recall the identity (27.9) in its new dress:

$$u(\bar{x}) - u(a) = \sum_{i=1}^{m} \frac{u(x_i) - u(x_{i-1})}{x_i - x_{i-1}}(x_i - x_{i-1}). \tag{27.12}$$

By the uniform differentiability of u:

$$u(x_i) - u(x_{i-1}) = u'(x_{i-1})(x_i - x_{i-1}) + E_u(x_i, x_{i-1}),$$

where

$$|E_u(x_i, x_{i-1})| \leq K_u(x_i - x_{i-1})^2, \tag{27.13}$$

with K_u a constant, we can write the identity as follows:

$$u(\bar{x}) - u(a) = \sum_{i=1}^{m} u'(x_{i-1})(x_i - x_{i-1}) + \sum_{i=1}^{m} E_u(x_i, x_{i-1}). \tag{27.14}$$

Setting h equal to the largest increment $x_i - x_{i-1}$, so that $x_i - x_{i-1} \leq h$ for all i, we find

$$\sum_{i=1}^{m} |E_u(x_i, x_{i-1})| \leq \sum_{i=1}^{m} K_u(x_i - x_{i-1})h = K_u(\bar{x} - a)h.$$

The formula (27.14) can thus be written

$$u(\bar{x}) - u(a) = \sum_{i=1}^{m} u'(x_{i-1})(x_i - x_{i-1}) + E_h, \tag{27.15}$$

where

$$|E_h| \le K_u(\bar{x} - a)h. \tag{27.16}$$

The Fundamental Theorem is the following analog of this formula:

$$u(\bar{x}) - u(a) = \int_a^{\bar{x}} u'(x)\, dx, \tag{27.17}$$

with the sum $\sum$ corresponding to the integral sign $\int$, the increments $x_i - x_{i-1}$ corresponding to dx, and x_i corresponding to the integration variable x. We see by (27.16) that the additional term E_h in (27.15) tends to zero as the maximal increment h tends to zero. We thus expect (27.17) to be a limit form of (27.15) as h tends to zero.

27.8 A Proof of the Fundamental Theorem of Calculus

We now give a full proof of the Fundamental theorem. We assume for simplicity that $[a, b] = [0, 1]$ and the initial value $u(0) = 0$. We comment on the general problem at the end of the proof. So the problem we consider is: Given a Lipschitz continuous function $f : [0, 1] \to \mathbb{R}$, find a solution $u(x)$ of the initial value problem,

$$\begin{cases} u'(x) = f(x) & \text{for } 0 < x \le 1, \\ u(0) = 0. \end{cases} \tag{27.18}$$

We shall now construct an approximation to the solution $u(x)$ and give a meaning to the solution formula

$$u(\bar{x}) = \int_0^{\bar{x}} f(x)\, dx \quad \text{for } 0 \le \bar{x} \le 1.$$

To this end, let n be a natural number and let $0 = x_0 < x_1 < \ldots < x_N = 1$ be the subdivision of the interval $[0, 1]$ with *nodes* $x_i^n = ih_n$, $i = 0, \ldots, N$, where $h_n = 2^{-n}$ and $N = 2^n$. We thus divide the given interval $[0, 1]$ into subintervals $I_i^n = (x_{i-1}^n, x_i^n]$ of equal lengths $h_n = 2^{-n}$, see Fig. 27.3.

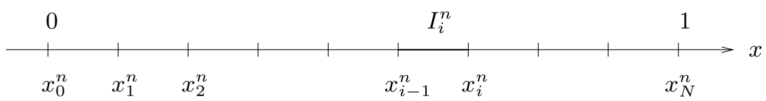

Fig. 27.3. Subintervals I_i^n of lengths $h_n = 2^{-n}$

The approximation to $u(x)$ is a continuous piecewise linear function $U^n(x)$ defined by the formula

$$U^n(x_j^n) = \sum_{i=1}^{j} f(x_{i-1}^n) h_n \qquad \text{for } j = 1, \ldots, N, \tag{27.19}$$

where $U^n(0) = 0$. This formula gives the values of $U^n(x)$ at the nodes $x = x_j^n$ and we extend $U^n(x)$ linearly between the nodes to get the rest of the values, see Fig. 27.4.

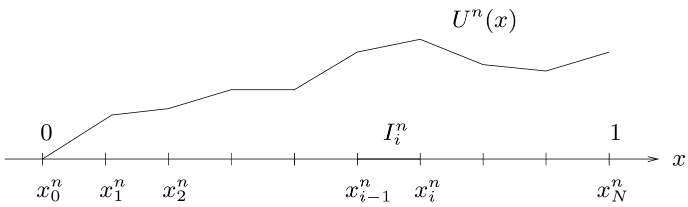

Fig. 27.4. Piecewise linear function $U^n(x)$

We see that $U^n(x_j^n)$ is a sum of contributions $f(x_{i-1}^n)h_n$ for all intervals I_i^n with $i \leq j$. By construction,

$$U^n(x_i^n) = U^n(x_{i-1}^n) + f(x_{i-1}^n)h_n \quad \text{for } i = 1, \ldots, N, \tag{27.20}$$

so given the function $f(x)$, we can compute the function $U^n(x)$ by using the formula (27.20) successively with $i = 1, 2, \ldots, N$, where we first compute $U^n(x_1^n)$ using the value $U^n(x_0^n) = U^n(0) = 0$, then $U^n(x_2^n)$ using the value $U^n(x_1^n)$ and so on. We may alternatively use the resulting formula (27.19) involving summation, which of course just amounts to computing the sum by successively adding the terms of the sum.

The function $U^n(x)$ defined by (27.19) is thus a continuous piecewise linear function, which is computable from the nodal values $f(x_i^n)$, and we shall now motivate why $U^n(x)$ should have a good chance of being an approximation of a function $u(x)$ satisfying (27.18). If $u(x)$ is uniformly differentiable on $[0, 1]$, then

$$u(x_i^n) = u(x_{i-1}^n) + u'(x_{i-1}^n)h_n + E_u(x_i^n, x_{i-1}^n) \quad \text{for } i = 1, \ldots, N, \tag{27.21}$$

where $|E_u(x_i^n, x_{i-1}^n)| \leq K_u(x_i^n - x_{i-1}^n)^2 = K_u h_n^2$, and consequently

$$u(x_j^n) = \sum_{i=1}^{j} u'(x_{i-1}^n)h_n + E_h \quad \text{for } j = 1, \ldots, N, \tag{27.22}$$

where $|E_h| \leq K_u h_n$, since $\sum_{i=1}^{j} h_n = jh_n \leq 1$. Assuming that $u'(x) = f(x)$ for $0 < x \leq 1$, the connection between (27.20) and (27.21) and (27.19) and (27.22) becomes clear considering that the terms $E_u(x_i^n, x_{i-1}^n)$ and E_h are small. We thus expect $U^n(x_j^n)$ to be an approximation of $u(x_j^n)$ at the nodes x_j^n, and therefore $U^n(x)$ should be an increasingly accurate approximation of $u(x)$ as n increases and $h_n = 2^{-n}$ decreases.

To make this approximation idea precise, we first study the convergence of the functions $U^n(x)$ as n tends to infinity. To do this, we fix $\bar{x} \in [0, 1]$

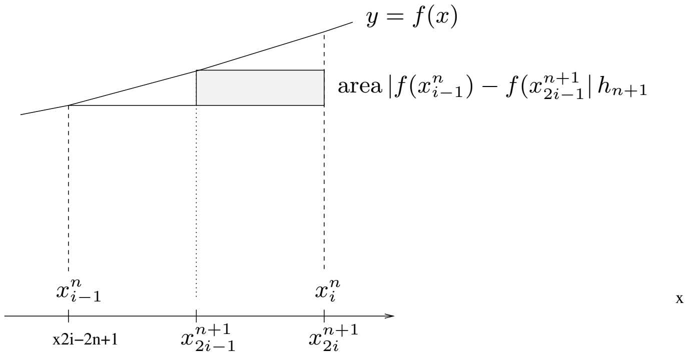

Fig. 27.5. The difference between $U^{n+1}(x)$ and $U^n(x)$

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and consider the sequence of numbers $\{U^n(\bar{x})\}_{n=1}^{\infty}$. We want to prove that this is a Cauchy sequence and thus we want to estimate $|U^n(\bar{x}) - U^m(\bar{x})|$ for $m > n$.

We begin by estimating the difference $|U^n(\bar{x}) - U^{n+1}(\bar{x})|$ for two consecutive indices n and $m = n + 1$. Recall that we used this approach in the proof of the Contraction Mapping theorem. We have

$$U^n(x_i^n) = U^n(x_{i-1}^n) + f(x_{i-1}^n)h_n,$$

and since $x_{2i}^{n+1} = x_i^n$ and $x_{2i-2}^{n+1} = x_{i-1}^n$,

$$\begin{aligned} U^{n+1}(x_i^n) &= U^{n+1}(x_{2i}^{n+1}) = U^{n+1}(x_{2i-1}^{n+1}) + f(x_{2i-1}^{n+1})h_{n+1} \\ &= U^{n+1}(x_{i-1}^n) + f(x_{2i-2}^{n+1})h_{n+1} + f(x_{2i-1}^{n+1})h_{n+1}. \end{aligned}$$

Subtracting and setting $e_i^n = U^n(x_i^n) - U^{n+1}(x_i^n)$, we have

$$e_i^n = e_{i-1}^n + (f(x_{i-1}^n)h_n - f(x_{2i-2}^{n+1})h_{n+1} - f(x_{2i-1}^{n+1})h_{n+1}),$$

that is, since $h_{n+1} = \frac{1}{2}h_n$,

$$e_i^n - e_{i-1}^n = (f(x_{i-1}^n) - f(x_{2i-1}^{n+1}))h_{n+1}. \tag{27.23}$$

Assuming that $\bar{x} = x_j^n$ and using (27.23) and the facts that $e_0^n = 0$ and $|f(x_{i-1}^n) - f(x_{2i-1}^{n+1})| \le L_f h_{n+1}$, we get

$$\begin{aligned} |U^n(\bar{x}) - U^{n+1}(\bar{x})| = |e_j^n| &= |\sum_{i=1}^{j}(e_i^n - e_{i-1}^n)| \\ &\le \sum_{i=1}^{j}|e_i^n - e_{i-1}^n| = \sum_{i=1}^{j}|f(x_{i-1}^n) - f(x_{2i-1}^{n+1})|h_{n+1} \\ &\le \sum_{i=1}^{j} L_f h_{n+1}^2 = \frac{1}{4}L_f h_n \sum_{i=1}^{j} h_n = \frac{1}{4}L_f \bar{x} h_n, \end{aligned} \tag{27.24}$$

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where we also used the fact that $\sum_{i=1}^{j} h_n = \bar{x}$. Iterating this estimate and using the formula for a geometric sum, we get

$$\begin{aligned}|U^n(\bar{x}) - U^m(\bar{x})| &\leq \frac{1}{4} L_f \bar{x} \sum_{k=n}^{m-1} h_k = \frac{1}{4} L_f \bar{x} (2^{-n} + \ldots + 2^{-m+1}) \\ &= \frac{1}{4} L_f \bar{x} 2^{-n} \frac{1 - 2^{-m+n}}{1 - 2^{-1}} \leq \frac{1}{4} L_f \bar{x} 2^{-n} 2 = \frac{1}{2} L_f \bar{x} h_n,\end{aligned}$$

that is

$$|U^n(\bar{x}) - U^m(\bar{x})| \leq \frac{1}{2} L_f \bar{x} h_n. \tag{27.25}$$

This estimate shows that $\{U^n(\bar{x})\}_{n=1}^{\infty}$ is a Cauchy sequence and thus converges to a real number. We decide, following Leibniz, to denote this real number by

$$\int_0^{\bar{x}} f(x)\, dx,$$

which thus is the limit of

$$U^n(\bar{x}) = \sum_{i=1}^{j} f(x_{i-1}^n) h_n$$

as n tends to infinity, where $\bar{x} = x_j^n$. In other words,

$$\int_0^{\bar{x}} f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{j} f(x_{i-1}^n) h_n.$$

Letting m tend to infinity in (27.25), we can express this relation in quantitative form as follows:

$$\left| \int_0^{\bar{x}} f(x)\, dx - \sum_{i=1}^{j} f(x_{i-1}^n) h_n \right| \leq \frac{1}{2} L_f \bar{x} h_n.$$

At this point, we have defined the integral $\int_0^{\bar{x}} f(x)\, dx$ for a given Lipschitz continuous function $f(x)$ on $[0,1]$ and a given $\bar{x} \in [0,1]$, as the limit of the sequence $\{U^n(\bar{x})\}_{n=1}^{\infty}$ as n tends to infinity. We can thus define a function $u : [0,1] \to \mathbb{R}$ by the formula

$$u(\bar{x}) = \int_0^{\bar{x}} f(x)\, dx \quad \text{for } \bar{x} \in [0,1]. \tag{27.26}$$

We now proceed to check that the function $u(x)$ defined in this way indeed satisfies the differential equation $u'(x) = f(x)$. We proceed in two steps. First we show that the function $u(x)$ is Lipschitz continuous on $[0,1]$ and then we show that $u'(x) = f(x)$.

Before plunging into these proofs, we need to address a subtle point. Looking back at the construction of $u(x)$, we see that we have defined $u(\bar{x})$ for $\bar{x}$ of the form $\bar{x} = x_j^n$, where $j = 0, 1, \dots, 2^n$, $n = 1, 2, \dots,$. These are the rational numbers with finite decimal expansion in the base of 2, and they are *dense* in the sense that for any real number $x \in [0, 1]$ and any $\epsilon > 0$, there is a point of the form x_j^n so that $|x - x_j^n| \leq \epsilon$. Recalling the Chapter *Real numbers*, we understand that if we can show that $u(x)$ is Lipschitz continuous on the dense set of numbers of the form x_j^n, then we can extend $u(x)$ as a Lipschitz function to the set of real numbers $[0, 1]$.

We thus assume that $\bar{x} = x_j^n$ and $\bar{y} = x_k^n$ with $j > k$, and we note that

$$U^n(\bar{x}) - U^n(\bar{y}) = \sum_{i=1}^{j} f(x_{i-1}^n)h_n - \sum_{i=1}^{k} f(x_{i-1}^n)h_n = \sum_{i=k+1}^{j} f(x_{i-1}^n)h_n$$

and using the triangle inequality

$$|U^n(\bar{x}) - U^n(\bar{y})| \leq \sum_{i=k+1}^{j} |f(x_{i-1}^n)|h_n \leq M_f \sum_{i=k+1}^{j} h_n = M_f|\bar{x} - \bar{y}|,$$

where M_f is a positive constant such that $|f(x)| \leq M_f$ for all $x \in [0, 1]$. Letting n tend to infinity, we see that

$$u(\bar{x}) - u(\bar{y}) = \int_0^{\bar{x}} f(x)\,dx - \int_0^{\bar{y}} f(x)\,dx = \int_{\bar{y}}^{\bar{x}} f(x)\,dx, \tag{27.27}$$

where of course,

$$\int_{\bar{y}}^{\bar{x}} f(x)\,dx = \lim_{n\to\infty} \sum_{i=k+1}^{j} f(x_{i-1}^n)h_n,$$

and also

$$|u(\bar{x}) - u(\bar{y})| \leq \left|\int_{\bar{y}}^{\bar{x}} f(x)\,dx\right| \leq \int_{\bar{y}}^{\bar{x}} |f(x)|\,dx \leq M_f|\bar{x} - \bar{y}|, \tag{27.28}$$

where the second inequality is the so-called *triangle inequality for integrals* to be proved in the next section. We thus have

$$|u(\bar{x}) - u(\bar{y})| \leq M_f|\bar{x} - \bar{y}|, \tag{27.29}$$

which proves the Lipschitz continuity of $u(x)$.

We now prove that the function $u(x)$ defined for $x \in [0, 1]$ by the formula

$$u(x) = \int_a^x f(y)\,dy,$$

where $f : [0,1] \to \mathbb{R}$ is Lipschitz continuous, satisfies the differential equation

$$u'(x) = f(x) \quad \text{for } x \in [0,1],$$

that is

$$\frac{d}{dx}\int_0^x f(y)\,dy = f(x). \tag{27.30}$$

To this end, we choose $x, \bar{x} \in [0,1]$ with $x \geq \bar{x}$ and use (27.27) and (27.28) to see that

$$u(x) - u(\bar{x}) = \int_0^x f(z)dz - \int_0^{\bar{x}} f(y)dy = \int_{\bar{x}}^x f(y)dy,$$

and

$$\begin{aligned}|u(x) - u(\bar{x}) - f(\bar{x})(x-\bar{x})| &= \Big|\int_{\bar{x}}^x f(y)\,dy - f(\bar{x})(x-\bar{x})\Big| \\ = \Big|\int_{\bar{x}}^x (f(y) - f(\bar{x}))\,dy\Big| &\leq \int_{\bar{x}}^x |f(y) - f(\bar{x})|\,dy \\ &\leq \int_{\bar{x}}^x L_f|y-\bar{x}|\,dy = \frac{1}{2}L_f(x-\bar{x})^2,\end{aligned}$$

where we again used the triangle inequality for integrals. This proves that u is uniformly differentiable on $[0,1]$, and that $K_u \leq \frac{1}{2}L_f$.

Finally to prove uniqueness, we recall from (27.15) and (27.16) that a function $u : [0,1] \to \mathbb{R}$ with Lipschitz continuous derivative $u'(x)$ and $u(0) = 0$, can be represented as

$$u(\bar{x}) = \sum_{i=1}^m u'(x_{i-1})(x_i - x_{i-1}) + E_h,$$

where

$$|E_h| \leq K_u(\bar{x} - a)h.$$

Letting n tend to infinity, we find that

$$u(\bar{x}) = \int_0^{\bar{x}} u'(x)\,dx \quad \text{for } \bar{x} \in [0,1], \tag{27.31}$$

which expresses the fact that a uniformly differentiable function with Lipschitz continuous derivative is the integral of its derivative. Suppose now that $u(x)$ and $v(x)$ are two uniformly differentiable functions on $[0,1]$ satisfying $u'(x) = f(x)$, and $v'(x) = f(x)$ for $0 < x \leq 1$, and $u(0) = u_0$, $v(0) = u_0$, where $f : [0,1] \to \mathbb{R}$ is Lipschitz continuous. Then the difference $w(x) = u(x) - v(x)$ is a uniformly differentiable function on $[0,1]$ satisfying $w'(x) = 0$ for $a < x \leq b$ and $w(0) = 0$. But we just showed that

$$w(x) = \int_a^x w'(y)\,dy,$$

and thus $w(x) = 0$ for $x \in [0,1]$. This proves that $u(x) = v(x)$ for $x \in [0,1]$ and the uniqueness follows.

Recall that we proved the Fundamental Theorem for special circumstances, namely on the interval $[0,1]$ with initial value 0. We can directly generalize the construction above by replacing $[0,1]$ by an arbitrary bounded interval $[a,b]$, replacing h_n by $h_n = 2^{-n}(b-a)$, and assuming instead of $u(0) = 0$ that $u(a) = u_a$, where u_a is a given real number. We have now proved the formidable Fundamental Theorem of Calculus.

Theorem 27.1 (Fundamental Theorem of Calculus) *If $f : [a,b] \to \mathbb{R}$ is Lipschitz continuous, then there is a unique uniformly differentiable function $u : [a,b] \to \mathbb{R}$, which solves the initial value problem*

$$\begin{cases} u'(x) = f(x) & \text{for } x \in (a,b], \\ u(a) = u_a, \end{cases} \tag{27.32}$$

where $u_a \in \mathbb{R}$ is given. The function $u : [a,b] \to \mathbb{R}$ can be expressed as

$$u(\bar{x}) = u_a + \int_a^{\bar{x}} f(x)\,dx \quad \text{for } \bar{x} \in [a,b],$$

where

$$\int_0^{\bar{x}} f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{j} f(x_{i-1}^n) h_n,$$

with $\bar{x} = x_j^n$, $x_i^n = a + ih_n$, $h_n = 2^{-n}(b-a)$. More precisely, if the Lipschitz constant of $f : [a,b] \to \mathbb{R}$ is L_f, then for $n = 1, 2, \ldots,$

$$\Big| \int_a^{\bar{x}} f(x)\,dx - \sum_{i=1}^{j} f(x_{i-1}^n) h_n \Big| \le \frac{1}{2}(\bar{x} - a) L_f h_n. \tag{27.33}$$

Furthermore if $|f(x)| \le M_f$ for $x \in [a,b]$, then $u(x)$ is Lipschitz continuous with Lipschitz constant M_f and $K_u \le \frac{1}{2} L_f$, where K_u is the constant of uniform differentiability of $u : [a,b] \to \mathbb{R}$.

27.9 Comments on the Notation

We can change the names of the variables and rewrite (27.26) as

$$u(x) = \int_0^x f(y)\,dy. \tag{27.34}$$

We will often use the Fundamental Theorem in the form

$$\int_a^b u'(x)\,dx = u(b) - u(a), \tag{27.35}$$

which states that the integral $\int_a^b f(x)\,dx$ is equal to the difference $u(b) - u(a)$, where $u(x)$ is a primitive function of $f(x)$. We will sometimes use the notation $[u(x)]_{x=a}^{x=b} = u(b) - u(a)$ or shorter $[u(x)]_a^b = u(b) - u(a)$, and write

$$\int_a^b u'(x)\,dx = \big[u(x)\big]_{x=a}^{x=b} = \big[u(x)\big]_a^b.$$

Sometimes the notation

$$\int f(x)\,dx,$$

without limits of integration, is used to denote a primitive function of $f(x)$. With this notation we would have for example

$$\int dx = x + C, \quad \int x\,dx = \frac{x^2}{2} + C, \quad \int x^2\,dx = \frac{x^3}{3} + C,$$

where C is a constant. We will not use this notation in this book. Note that the formula $x = \int dx$ may be viewed to express that "the whole is equal to the sum of the parts".

27.10 Alternative Computational Methods

Note that we might as well compute $U^n(x_i^n)$ from knowledge of $U^n(x_{i-1}^n)$, using the formula

$$U^n(x_i^n) = U^n(x_{i-1}^n) + f(x_i^n)h_n, \tag{27.36}$$

obtained by replacing $f(x_{i-1}^n)$ by $f(x_i^n)$, or

$$U^n(x_i^n) = U^n(x_{i-1}^n) + \frac{1}{2}(f(x_{i-1}^n) + f(x_i^n))h_n \tag{27.37}$$

using the mean value $\frac{1}{2}(f(x_{i-1}^n) + f(x_i^n))$. These alternatives may bring certain advantages, and we will return to them in Chapter *Numerical quadrature*. The proof of the Fundamental Theorem is basically the same with these variants and by uniqueness all the alternative constructions give the same result.

27.11 The Cyclist's Speedometer

An example of a physical situation modeled by the initial value problem (27.2) is a cyclist biking along a straight line with $u(x)$ representing the position at time x, $u'(x)$ being the speed at time x and specifying the position $u(a) = u_a$ at the initial time $x = a$. Solving the differential equation (27.2) amounts to determining the position $u(x)$ of the cyclist at time $a < x \leq b$, after specifying the position at the initial time $x = a$ and knowing the speed $f(x)$ at each time x. A standard bicycle speedometer may be viewed to solve this problem, viewing the speedometer as a device which measures the instantaneous speed $f(x)$, and then outputs the total traveled distance $u(x)$. Or is this a good example? Isn't it rather so that the speedometer measures the traveled distance and then reports the momentary (average) speed? To answer this question would seem to require a more precise study of how a speedometer actually works, and we urge the reader to investigate this problem.

27.12 Geometrical Interpretation of the Integral

In this section, we interpret the proof of the Fundamental Theorem as saying that the integral of a function is the area underneath the graph of the function. More precisely, the solution $u(\bar{x})$ given by (27.3) is equal to the area under the graph of the function $f(x)$ on the interval $[a, \bar{x}]$, see Fig. 27.6. For the purpose of this discussion, it is natural to assume that $f(x) \geq 0$.

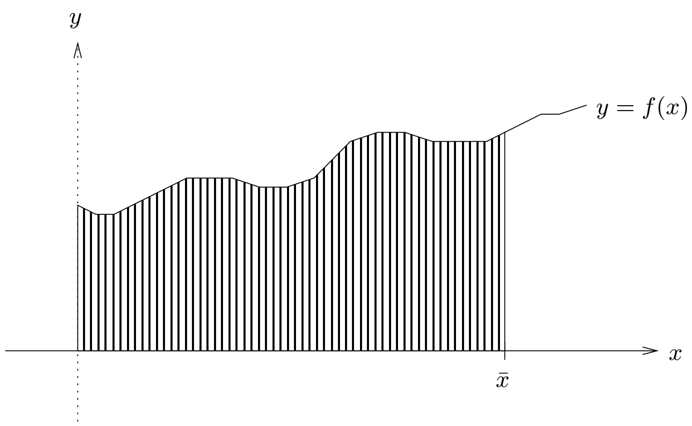

Fig. 27.6. Area under $y = f(x)$

Of course, we also have to explain what we mean by the area under the graph of the function $f(x)$ on the interval $[a, \bar{x}]$. To do this, we first interpret the approximation $U^n(\bar{x})$ of $u(\bar{x})$ as an area. We recall from the previous section that

$$U^n(x_j^n) = \sum_{i=1}^{j} f(x_{i-1}^n) h_n,$$

where $x_j^n = \bar{x}$. Now, we can view $f(x_{i-1}^n)h_n$ as the area of a rectangle with base h_n and height $f(x_{i-1}^n)$, see Fig. 27.7.

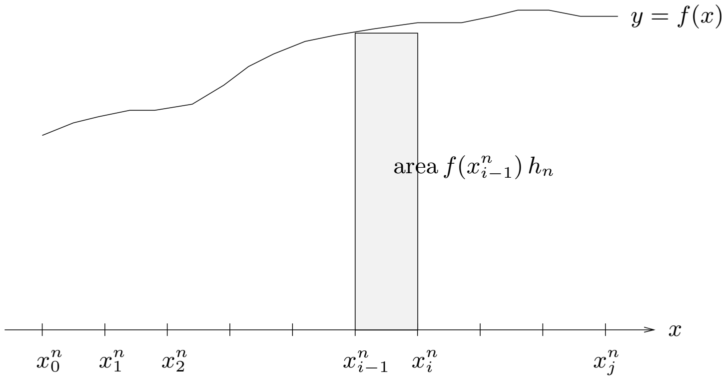

Fig. 27.7. Area $f(x_{i-1}^n)\,h_n$ of rectangle

We can thus interpret the sum

$$\sum_{i=1}^{j} f(x_{i-1}^n)h_n$$

as the area of a collection of rectangles which form a staircase approximation of $f(x)$, as displayed in Fig. 27.8. The sum is also referred to as a *Riemann sum*.

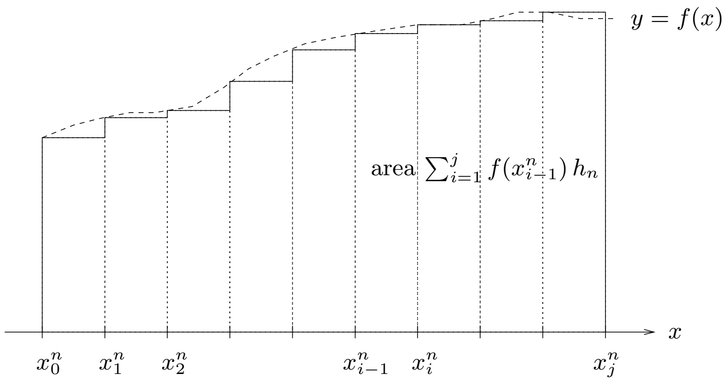

Fig. 27.8. Area $\sum_{i=1}^{j} f(x_{i-1}^n)\,h_n$ under a staircase approximation of $f(x)$

Intuitively, the area under the staircase approximation of $f(x)$ on $[a, \bar{x}]$, which is $U^n(\bar{x})$, will approach the area under the graph of $f(x)$ on $[a, \bar{x}]$ as n tends to infinity and $h_n = 2^{-n}(b-a)$ tends to zero. Since $\lim_{n\to\infty} U^n(\bar{x}) =$

$u(\bar{x})$, this leads us to *define* the area under $f(x)$ on the interval $[0, \bar{x}]$ as the limit $u(\bar{x})$.

Note the logic used here: The value $U^n(\bar{x})$ represents the area under a staircase approximation of $f(x)$ on $[a, \bar{x}]$. We know that $U^n(\bar{x})$ tends to $u(\bar{x})$ as n tends to infinity, and on intuitive grounds we feel that the limit of the area under the staircase should be equal to the area under the graph of $f(x)$ on $[a, \bar{x}]$. We then simply define the area under $f(x)$ on $[a, \bar{x}]$ to be $u(\bar{x})$. By definition we thus interpret the integral of $f(x)$ on $[0, \bar{x}]$ as the area under the graph of the function $f(x)$ on $[a, \bar{x}]$. Note that *this is an interpretation*. It is not a good idea to say the integral *is* an area. This is because the integral can represent many things, such as a distance, a quantity of money, a weight, or some thing else. If we interpret the integral as an area, then we also interpret a distance, a quantity of money, a weight, or some thing else, as an area. We understand that we cannot take this interpretation to be literally true, because a distance cannot *be equal* to an area, but it can be *interpreted* as an area. We hope the reader gets the (subtle) difference.

As an example, we compute the area A under the graph of the function $f(x) = x^2$ between $x = 0$ and $x = 1$ as follows

$$A = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{1}{3}.$$

This is an example of the magic of Calculus, behind its enormous success. We were able to compute an area, which in principle is the sum of very many very small pieces, without actually having to do the tedious and laborious computation of the sum. We just found a primitive function $u(x)$ of x^2 and computed $A = u(3) - u(0)$ without any effort at all. Of course we understand the telescoping sum behind this illusion, but if you don't see this, you must be impressed, right? To get a perspective and close a circle, we recall the material in Leibniz' teen-age dream in Chapter *A very short course in Calculus*.

27.13 The Integral as a Limit of Riemann Sums

The Fundamental Theorem of Calculus states that the integral of $f(x)$ over the interval $[a, b]$ is equal to a limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{2^n} f(x_{i-1}^n)h_n,$$

where $x_i^n = a + ih_n$, $h_n = 2^{-n}(b - a)$, or more precisely, for $n = 1, 2, \ldots,$

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{2^n} f(x_{i-1}^n)h_n \right| \le \frac{1}{2}(b - a)L_f h_n,$$ TS[j]

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where L_f is the Lipschitz constant of f. We can thus define the integral $\int_a^b f(x)\,dx$ as a limit of Riemann sums without invoking the underlying differential equation $u'(x) = f(x)$. This approach is useful in defining integrals of functions of several variables (so-called multiple integrals like double integrals and triple integrals), because in these generalizations there is no underlying differential equation.

In our formulation of the Fundamental Theorem of Calculus, we emphasized the coupling of the integral $\int_a^x f(y)\,dy$ to the related differential equation $u'(x) = f(x)$, but as we just said, we could put this coupling in the back-ground, and define the integral as a limit of Riemann sums without invoking the underlying differential equation. This connects with the idea that the integral of a function can be interpreted as the area under the graph of the function, and will find a natural extension to multiple integrals in Chapters *Double integrals* and *Multiple integrals*.

Defining the integral as a limit of Riemann sums poses a question of uniqueness: since there are different ways of constructing Riemann sums one may ask if all limits will be the same. We will return to this question in Chapter *Numerical quadrature* and (of course) give an affirmative answer.

27.14 An Analog Integrator

James Thompson, brother of Lord Kelvin, constructed in 1876 an analog mechanical integrator based on a rotating disc coupled to a cylinder through another orthogonal disc adjustable along the radius of the first disc, see Fig. 27.9.TS[k] The idea was to get around the difficulties of realizing the Analytical Engine, the mechanical digital computer envisioned by Babbage in the 1830s. Lord Kelvin tried to use a system of such analog integrators to compute different problems of practical interest including that of tide prediction, but met serious problems to reach sufficient accuracy. Similar ideas ideas were taken up by Vannevar Bush at MIT Massachussetts Institute of Technology in the 1930s, who constructed a *Differential Analyzer* consisting of a collection of analog integrators, which was programmable to solve differential equations, and was used during the Second World War for computing trajectories of projectiles. A decade later the digital computer took over the scene, and the battle between arithmetic and geometry initiated between the Pythagorean and Euclidean schools more than 2000 years ago, had finally come an end.

TS[k] In the hardcopy, is here a reference of Fig. 27.10, please check it.

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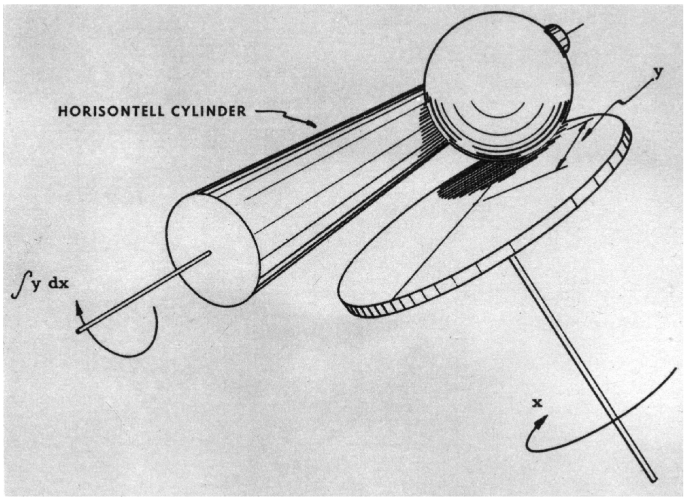

Fig. 27.9. The principle of an Analog Integrator

Chapter 27 Problems

27.1. Determine primitive functions on $\mathbb{R}$ to (a) $(1+x^2)^{-2}2x$, (b) $(1+x)^{-99}$, (c) $(1+(1+x^3)^2)^{-2}2(1+x^3)3x^2$.

27.2. Compute the area under the graph of the function $(1+x)^{-2}$ between $x=1$ and $x=2$.

27.3. A car travels along the x-axis with speed $v(t)=t^{\frac{3}{2}}$ starting at $x=0$ for $t=0$. Compute the position of the car for $t=10$.

27.4. Carry out the proof of the Fundamental Theorem for the variations (27.36) and (27.37).

27.5. Construct a *mechanical integrator* solving the differential equation $u'(x)=f(x)$ for $x>0$, $u(0)=0$ through an analog mechanical devise. Hint: Get hold of a rotating cone and a string.

27.6. Explain the principle behind Thompson's analog integrator.

27.7. Construct a *mechanical* speedometer reporting the speed and traveled distance. Hint: Check the construction of the speedometer of your bike.

27.8. Find the solutions of the initial value problem $u'(x)=f(x)$ for $x>0$, $u(0)=1$, in the following cases: (a) $f(x)=0$, (b) $f(x)=1$, (c) $f(x)=x^r,\ r>0$.

27.9. Find the solution to the second order initial value problem $u''(x)=f(x)$ for $x>0$, $u(0)=u'(0)=1$, in the following cases: (a) $f(x)=0$, (b) $f(x)=1$, (c) $f(x)=x^r,\ r>0$. Explain why two initial conditions are specified.

27.10. Solve initial value problem $u'(x) = f(x)$ for $x \in (0, 2]$, $u(0) = 1$, where $f(x) = 1$ for $x \in [0, 1)$ and $f(x) = 2$ for $x \in [1, 2]$. Draw a graph of the solution and calculate $u(3/2)$. Show that $f(x)$ is not Lipschitz continuous on $[0, 2]$ and determine if $u(x)$ is Lipschitz continuous on $[0, 2]$.

27.11. The time it takes for a light beam to travel through an object is $t = \frac{d}{c/n}$, where c is the speed of light in vacuum, n is the refractive index of the object and d is its thickness. How long does it take for a light beam to travel the shortest way through the center of a glass of water, if the refractive index of the water varies as a certain function $n_w(r)$ with the distance r from the center of glass, the radius of the glass is R and the thickness and that the walls have constant thickness h and constant refractive index equal to n_g.

27.12. Assume that f and g are Lipschitz continuous on $[0, 1]$. Show that $\int_0^1 |f(x) - g(x)|dx = 0$ if and only if $f = g$ on $[0, 1]$. Does this also hold if $\int_0^1 |f(x) - g(x)|dx$ is replaced by $\int_0^1 (f(x) - g(x))dx$?

Fig. 27.10. David Hilbert (1862–1943) at the age of 24: "A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street"