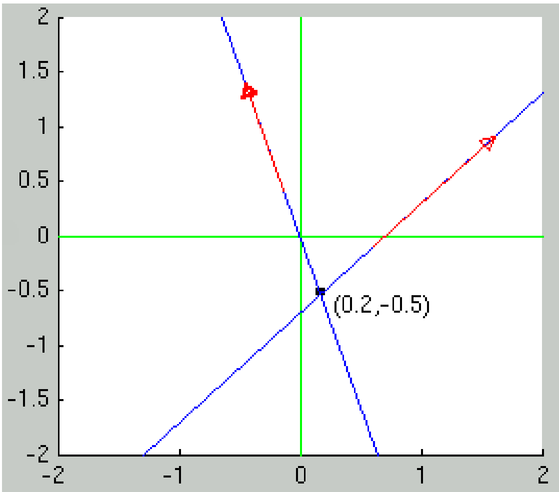
LINEAR SYSTEMS

COMPUTER SESSION A6



BACKGROUND

Many engineering problems give rise to a system of linear equations (a linear system) of the form:

$$Ax = b$$

where A is a matrix and x and b are vectors. A and b are known and x is unknown. Such a system with two equations looks like this with the components expanded:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

If we perform the matrix-vector multiplication, we get:

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

or, perhaps in a more familiar form (moving the x_2 term to the right and dividing by the coefficient in front of x_1:

$$\begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 \\ \frac{b_2}{a_{21}} - \frac{a_{22}}{a_{21}}x_2 \end{pmatrix}$$

Thus, the equation system $Ax = b$ of two equations can be viewed as two lines. The solution of the equation system is the point where the lines intersect.

We see that there is a geometric interpretation of linear equation systems, and vice versa; it's possible to solve geometric problems using linear equation systems. This geometric interpretation can help with the understanding of linear systems.

Before today's computer session, make sure that you understand and can answer the following questions.

Question 1

What is a vector? What is a matrix? How do you define a vector and matrix in Matlab?

Question 2

Perform the matrix-vector multiplication above and verify that you get the same answer.

PREPARATIONS

The session is divided into two parts. The first part involves experimenting in the Mathematics Laboratory and the second part involves trying out expressions and functions in Matlab and writing your own functions.

PREPARATIONS

The session is divided into two parts. The first part involves experimenting in the Mathematics Laboratory and the second part involves programming with vectors and matrices.

Start Matlab.

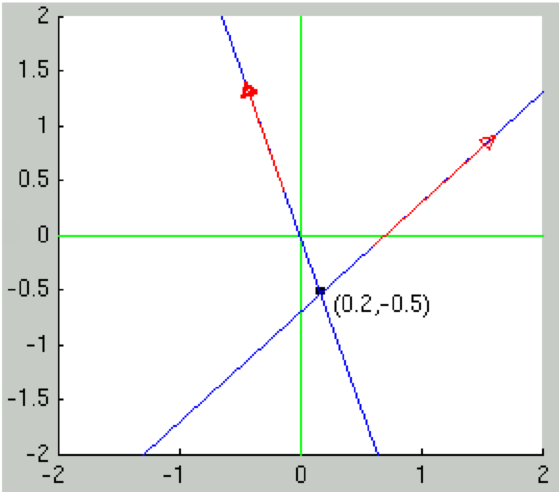
If you are working on the computers of the School of Chemical Engineering at Chalmers, then download the file `startmath.m` to your Matlab work directory (if you have not done this already). This file is available on the web page of this session under Programs and templates. Then type `startmath` at the Matlab prompt. This command sets the search path to the directories where the Mathematics Laboratory is kept.

If you are working on another computer, then download the file `MathematicsLaboratory.zip` to your Matlab work directory. This file is available on the web page of this session under Programs and templates. Unzip the file, it should create a directory guis in your Matlab work directory. At the Matlab prompt, type: `addpath guis`. You are now ready to use the Mathematics Laboratory.

Keep your AMBS book with you and open at the relevant chapters.

PROBLEMS

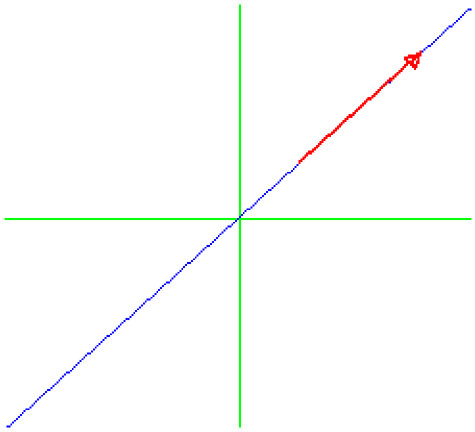
Problem 1 - Linear Systems in the Mathematics Laboratory.



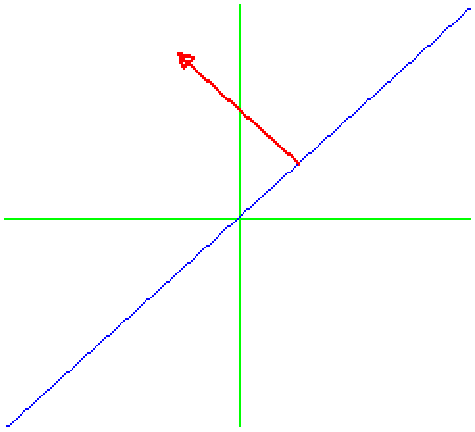
(1) Give the command `open('RM+.fig')` to open the Road Map to the Mathematics Laboratory, and press the Euclids P button to step out in the Euclidean plane. Alternatively you may enter this plane directly directly from the matlab prompt by the command `open('EUCLID.fig')`.

(2) Click on the clear button. Then click at the coords button, and study how the coordinates of the cursor point changes as you move the cursor around in the left window.

(3) Define a line in the plane by giving first a point on the line (in the `x` − edit box), and then a vector giving the direction of the line (in the // to edit box), followed by a return. A line should now appear with the given directional vector in red starting from the given point on the line. You can then edit the given line by point dragging (with the left mouse button down), on a point near the given point on the line to translate the line, or a point near the arrow head to rotate the line. Watch how the equation defining the line changes as you redefine the line.



The red arrow is a tangent of the blue line.



The red arrow is a normal of the blue line.

(4) Now try defining a line by pressing the click line button and then click at two points of the desired line. If there is no “handle” (directional vector) to the new line, you can get one by pressing the handles (on) button. You can also shift between a parallell vector or an orthogonal vector defining the direction of the line using the normal/tangent button.

With two lines present on the screen you should also see a black (or possibly blue) spot with given coordinates where the two lines intersect. This point represents the solution of the corresponding two linear equations displayed in the top right window, which you can toggle between vector form and component form. The vector form of this system of equation is also illustrated graphically in the lower right window, where the two vectors multiplied by x1 and x2, respectively, are colored in grey, and the target vector on the right is colored black. You can not edit these vectors directly, but study how they change as you change the two lines to the left.

Make sure you understand what is happening and the (rather intricate) interrelation between the three different representations, the formulas/equations in the top right window, the lines to the left, and the vectors to the lower right.

(5) You may solve the system of equations in the top right window several ways, by hand calculation using variable elimination, just by looking (at the coordinates of the intersection point) in the left window, or (as a sport) by seeking the combination of the two grey column arrows, colouered black, that matches the target column vector to the right, coloured purple. To do this you should first press the lines off button and then the trial solution button. You then move the cursor around with the left mouse button down in the left window until the corresponding black combination of the two grey column arrows matches the purple target vector of the right hand side of the system of equations. To check your answer you may then press the solve equations button.

Now for some exercises.

(6) Delete all but one line and change this by translation and by rotation, respectively. In what way does the equation for the line change as you a) translate b) rotate the line. What is the relation between the orthogonal arrow handle to a line and the coefficients of the corresponding equation?

(7) Create two lines which intersect in the point $(0.6, -.3)$.

(8) In which point do the lines `x - [ 0, 1] // [ 1, -1]` and `x - [ -1, -1] // [ 1, 0.5]` intersect?

(9) What is the corresponding $ax + by = c$ representation of these two lines? Are these representations unique, for given lines? Again, what is the relation between the equations in the top right window and the "handles" defining the line directions/normals?

(10) Solve the system of equations by hand calculation. Does your solution match the coordinates of the intersection point? Why/why not?

(11) Under what condition do two lines (in a plane) intersect?

(12) Can two lines in a plane have more than one intersection point?

(13) Now create a third line. Is there a common intersection point of the three lines? Could it be? Is it most natural to have or not to have such a common intersection point of more than two lines?

(14) How do you interpret the black spot with coordinates in the left window in the present case with more lines than two/more conditions/equations than variables to satisfy the equations? You may also try solving the corresponding set of three equations by hand, or by trying to combine the two grey column vectors to a given target

in three dimensional space! What is the reason the combination can seem to match the target, but yet miss?

Problem 2 - Linear systems in Matlab expressions.

(1) Create the 2×2 matrix A with entries `A(1, 1) = 1`, `A(1, 2) = 2`, `A(2, 1) = 3` and `A(2, 2) = 4`.
(2) Define vectors `e1 = [1; 0]` and `e2 = [0; 1]`. Compute `A * e1` and `A * e2`. Explain why `A * e1` equals the first column of A, and `A * e2` the second.
(3) Compute, using your "head computer", possibly supported by pen and paper, the product `A * v` with `v = [1; -2]`. Then check your answer using matlab.
(4) Does the equation $Ax = b$ have a (unique) solution (vector) `x = [x1; x2]` for any given vector `b = [b1; b2]`? Motivate your answer. Hint: recall a lecture about $a_1 \times a_2$ (cross product of the columns of A) being nonzero.
(5) Find (using pen and paper calculation) the solution of $Ax = b$ with `b = [6; 10]`. Then check your answer by computing `A * x` using matlab.
(6) Note that Ax in 5 equals Av in 3 scaled by the factor -2. Is there a corresponding relation between x and v ? Is this a coincidence or not? Hint: recall a lecture about $f(x) = Ax$ being linear.
(7) With A and b as in 5, compute `A \ b` (the matlab "backslash"). What do you get? (hint: compare to x in 5)
(8) Now define the matrix `C = [1 2; 3 6]`. Seek x such that $Cx = d$ with a) `d = [2; 3]` b) `d = [1.5; 4.5]`. Conclusions? How does matlab handle these equations, using `C \ d`?
(9) Compute AC (matrix-matrix multiplication). Compare with matlab. Is $AC = CA$? Again, check with matlab?
(10) Create the 2×3 - matrix B with entries `B(i, j) = 2 * i - j` in row i and column j. Compute AB and try to compute BA. Conclusions?
(11) Create the matrix `D = [A; C]`. Then check `size(D)`, `size(D, 1)` and `size(D, 2)`. If you are uncertain about the interpretation, consult `help size`. What is the size of DB? Check by computing `D * B`!
(12) Compute `A'`, `B'`, `D'` and `D''`. `A'` is called the transpose of A. How is `A'` denoted in AMBS? Check if `[1 2] == [1, 2]`. Check if `[1 2] == [1; 2]`. Check if `[1 2] == [1; 2]'`.

(13) Try the special commands `zeros(m, n)` and `ones(m, n)` with `m = 2` and `n = 3`, say. Define `F(2, 3) = 7`. What is `F(i, j)` for other combinations of i and j than $(2, 3)$? Also, try `eye(n)`, with `n = 2` and 3 say. Define `I = eye(2)`.

(14) Seek a matrix X such that $AX = I$. First try the following method. Given A as before, define

```
X = [A(2, 2) - A(1, 2); -A(2, 1) A(1, 1)] / \
    (A(1, 1)*A(2, 2) - A(1, 2)*A(2, 1))
```

Check that $AX = I$ by computing `A * X`. Check if also `X * A = I`. What is the matrix X called, and how is it denoted in this case?

(15) Compute `Y = A \ I` and compare to X.

(16) Solve `A * x1 = e1` and `A * x2 = e2` and define `Z = [x1 x2]`. How does Z compare to X and Y ? Do you see why ?

(17) Define vectors `a = [1 2 3]` and `b = [4 5 6]`. Compute the vector (cross) product $a \times b$.

(18) Compare the result in 17 to writing `cross(a, b)`. Compare `cross(a, b)` to `cross(b, a)`.

(19) Compute the scalar product $a{\cdot}b$ of a and b, first as `a(1) * b(1) + ..`, then as `sum(a .* b)`, then as `a * b'`, then as `dot(a, b)`. Make sure you understand how each of these alternatives work.

(20) Compute the projection of a onto b, and the projection of b onto a, respectively.

(21) What is the area of the triangle with vertices in the origin and in $(2, 1)$ and in $(-2, 3)$? Check your answer is reasonable by drawing a figure and estimating the area.

(22) What is the length (norm) $\|v\|$ of the vector `v = [-3 4]`? Compute both by pen and paper and using matlab. Recall that $\|v\| = \sqrt{v \cdot v}$. Also, compare to `norm(v)`. What does `length(v)` give in matlab? Check `help length` (and recall the function `size()`).

(23) What is the angle between the vectors a) v and `w = [8 6]` b) v and `u = [2 1]`? (hint: $v \cdot w = \|v\|\|w\|cos(x)$ so that $cos(x) = \frac{v \cdot w}{\|v\|\|w\|}$). In b) you may solve this equation using your Bisection solver, or check out `help acos`.

(24) Try the following matlab function for drawing a vector `a = (a(1), a(2))`.

```
function arrow(a)
% Draws an arrow representing the vector a
  n = [-a(2), a(1)];
  X = [0, a(1), .9 * a(1) + \
    .05 * n(1), a(1), .9 * a(1) - .05 * n(1)];
```

```
  Y = [0, a(2), .9 * a(2) + \
    .05 * n(2), a(2), .9 * a(2) - .05 * n(2)];

  % The following are Matlab graphics functions, they are
  % not part of this course.

  line(X, Y);
  set(gca, 'XLim', [-1, 1] * max(abs(X)), \
  'YLim', [-1, 1] * max(abs(Y)));
  axis('equal');
```

Save as `arrow.m`. Then try it out for different vectors a, for example `arrow([2 1])` and then `arrow([-1 2])`. What do you then see?

(25) Maybe you then want to make a vector move around, for example by a matlab function of the form:

```
function CounterClock()
% Animation of a rotating vector (arrow)

  % The following are Matlab graphics functions, they are
  % not part of this course.

  figure(1);
  set(gcf, 'DoubleBuffer', 'on');
  for v = 0 : .1 : 10
    a = [cos(v), sin(v)];
    arrow(a);
    set(gca, 'XLim', [-2 2], 'YLim', [-2 2]);
    drawnow;
    cla;
  end
```

SOLUTIONS

Make sure that you really try to solve each problem before looking at the solutions. Have you really tried to solve the problem or should you try again before looking at the solution?

The solutions are available on the web page of this session under *Solutions to problems*.

About

This Computer Session is part of the Body and Soul educational program. More information can be found at

```
http://www.phi.chalmers.se/bodysoul/
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