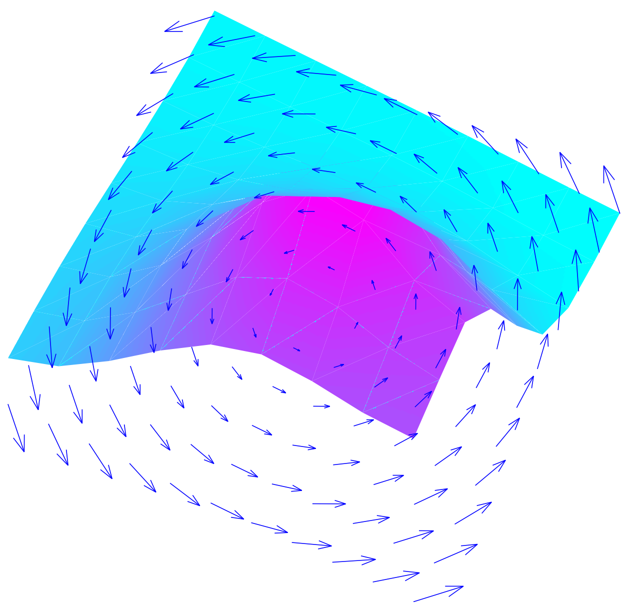
Convection–diffusion

Computer Session E3



Background

Today, we will solve the time-dependent convection–diffusion equation,

$$\begin{aligned} \dot{u} + b \cdot \nabla u - \nabla \cdot (a \nabla u) &= f \quad \text{in } \Omega \times (0,T], \\ -a\partial_n u &= \gamma(u - g_D) + g_N \quad \text{on } \Gamma \times (0,T], \\ u(\cdot,0) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where the convection is given by the vector $b = b(x,t)$ and the diffusion is given by $a = a(x,t)$. The dG(0) formulation of the convection–diffusion equation is given by

$$\begin{aligned}\int_\Omega U_n v\,dx + k\int_\Omega (b\cdot\nabla U_n\,v + a\nabla U_n\cdot\nabla v)\,dx + k\int_\Gamma \gamma U_n v\,ds =\\ k\int_\Omega f_n v\,dx + k\int_\Gamma (\gamma g_D - g_N)v\,ds + \int_\Omega U_{n-1} v\,dx \quad \forall v\in V_h,\end{aligned} \tag{2}$$

where $k = t_n - t_{n-1}$ denotes the size of the time step, U_n denotes the value at $t = t_n$, and U_{n-1} denotes the value at $t = t_{n-1}$. Note that also γ, g_D, and g_N should be evaluated at $t = t_n$. The dG(0) method is also known as the backward Euler method.

Before today's computer session, make sure that you understand and can answer the following questions.

Question 1 Derive the variational formulation (2) from the convection–diffusion equation (1).

Question 2 How does the corresponding variational formulation look for the cG(1) method? This method is also known as the Crank-Nicolson method.

Question 3 Verify that

$$b(x,t) = (-(x_2 - c_2), x_1 - c_1), \tag{3}$$

where $c = (c_1, c_2)$ is a constant, is divergence-free, i.e., $\nabla\cdot b = 0$. Draw a simple sketch of the vector-field given by b. (Draw some arrows on a piece of paper.) Why is it important that b is divergence-free?

Preparations

Create a new directory called `e3` and the two subdirectories `problem1` and `problem2`. Then download the following files to each of the subdirectories:

- `AssembleMatrix.m`,
- `AssembleVector.m`,

- `ConvDiff.m`,
- the files in the directory `mesh`.

These files are available on the web page of this session under *Programs and templates.*

Problems

Problem 1

Preparation

Go to the directory `problem1`. Create a file called `ConvDiffSolver.m` and write your program in this file. You also have to edit the file `ConvDiff.m`, where you specify the variational formulation.

Problem

Solve the convection–diffusion equation on the unit square (`square.m`) with homogeneous Neumann boundary conditions, $a = 0.1$, $b = (-2, 0)$, initial condition $u_0 = 0$, and $T = 3.5$, using a time step of size $k = 0.05$.

Let the source term (the right-hand side) be given by

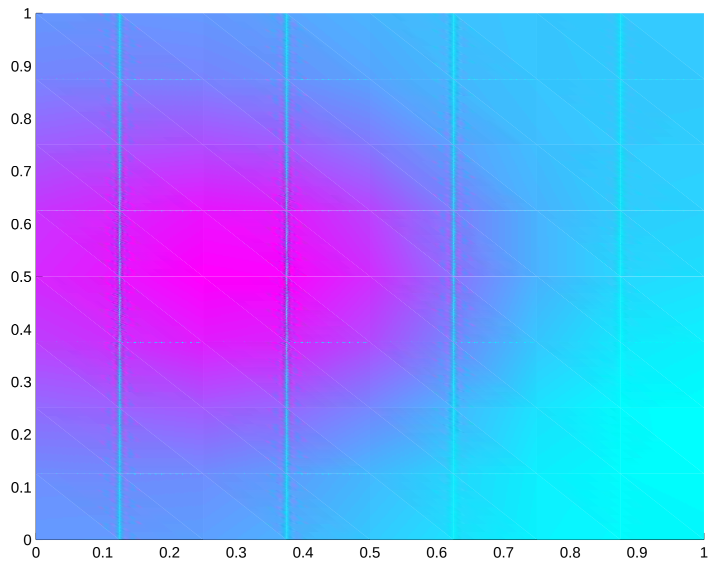
$$f(x,t) = \begin{cases} 1, & \text{if } |x - (0.75, 0.5)| < 0.1 \text{ and } |t - \text{round}(t)| < 0.1, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

corresponding to a new drop being added at $x = (0.75, 0.5)$ every second.

To solve the problem, look at your solver for the heat equation from computer session E2, or try yourself without looking.

Plot the solution using the `pdesurf` command in each time step. Remember to write `pause` after each plot.

Check your answer: Your solution should increase close to $x = (0.75, 0.5)$ every whole second (including $t = 0$). It should then drift off in the direction of the convection to the left, and be gradually flattened out by the diffusion, until a new drop is added at the next whole second.



Problem 2

Preparation

Go to the directory `problem2` and copy the two files `ConvDiffSolver.m` and `ConvDiff.m` from the directory `problem1`.

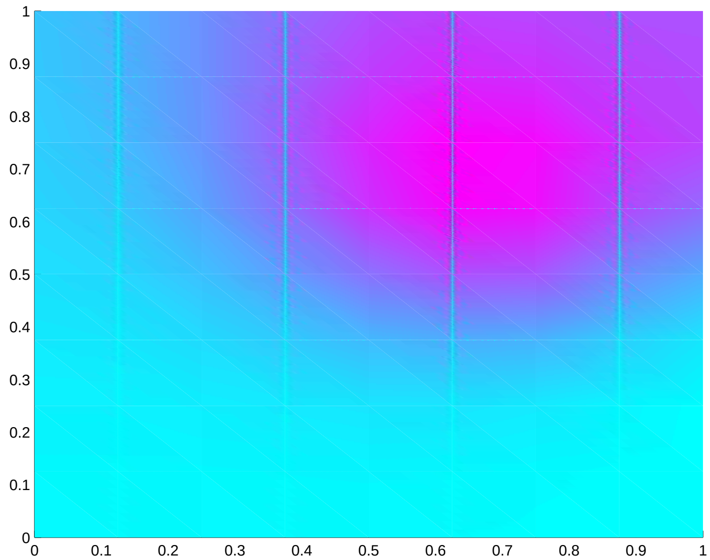
Problem

Now modify your program from Problem 1 to use the convection

$$b(x, t) = 5(-(x_2 - 0.5), x_1 - 0.5). \tag{5}$$

Create a movie of your solution using the commands `getframe` and `movie`.

Check your answer: Your solution should increase close to $x = (0.75, 0.5)$ every whole second (including $t = 0$). It should then drift off counter-clockwise in the domain in the direction of the convection, and be gradually flattened out by the diffusion, until a new drop is added at the next whole second.



Hints

Problem 1

The variational formulation that you need to put in the file `ConvDiff.m` is

```
u*v*dx + k*(0.1*du'*dv*dx + b(x,d,t)'*du*v*dx + g(x,d,t)*u*v*ds)
```

for the left-hand side and

```
k*(f(x,d,t)*v*dx + (g(x,d,t)*gd(x,d,t) - gn(x,d,t))*v*ds) + w*v*dx
```

for the right-hand side.

Problem 2

Use the commands `help movie` or `help getframe` to find out how to create a movie.

Solutions

Make sure that you really try to solve each problem before looking at the solutions. Have you really tried to solve the problem or should you try again before looking at the solution?

The solutions are available on the web page of this session under *Solutions to problems*.

About

This Computer Session is part of the Body and Soul educational program. More information can be found at

`http://www.phi.chalmers.se/bodysoul/`

This Computer session is written by Anders Logg.