Volume 1

Derivatives and Geometry in $\mathbb{R}^3$

$$|u(x_j) - u(x_{j-1})| \leq L_u |x_j - x_{j-1}|$$

$$u(x_j) - u(x_{j-1}) \approx u'(x_{j-1})(x_j - x_{j-1})$$

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

1
What is Mathematics?

> The question of the ultimate foundations and the ultimate meaning of mathematics remains open; we do not know in what direction it will find its final solution or whether a final objective answer may be expected at all. "Mathematizing" may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization. (Weyl)

1.1 Introduction

We start out by giving a very brief idea of the nature of mathematics and the role of mathematics in our society.

1.2 The Modern World: Automatized Production and Computation

The mass consumption of the *industrial society* is made possible by the *automatized mass production* of material goods such as food, clothes, housing, TV-sets, CD-players and cars. If these items had to be produced by hand, they would be the privileges of only a select few.

Analogously, the emerging *information society* is based on mass consumption of *automatized computation* by computers that is creating a new "virtual reality" and is revolutionizing technology, communication, admin-

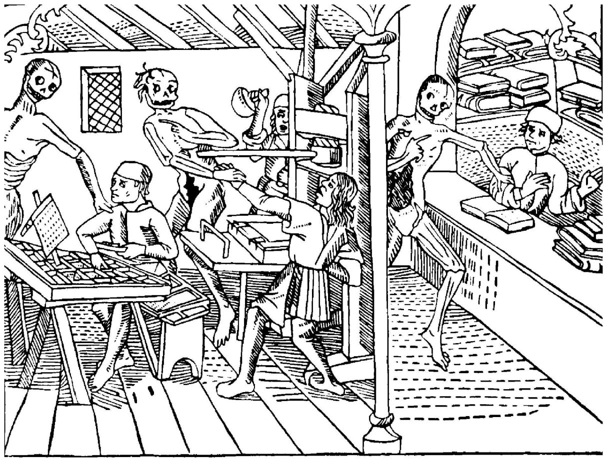

Fig. 1.1. First picture of book printing technique (from Danse Macabre, Lyon 1499)

istration, economy, medicine, and the entertainment industry. The information society offers immaterial goods in the form of knowledge, information, fiction, movies, music, games and means of communication. The modern PC or lap-top is a powerful computing device for mass production/consumption of information e.g. in the form of words, images, movies and music.

Key steps in the automatization or mechanization of production were: Gutenbergs's book printing technique (Germany, 1450), Christoffer Polhem's automatic machine for clock gears (Sweden, 1700), The Spinnning Jenny (England, 1764), Jacquard's punched card controlled weaving loom (France, 1801), Ford's production line (USA, 1913), see Fig. 1.1, Fig. 1.2, and Fig. 1.3.

Key steps in the automatization of computation were: Abacus (Ancient Greece, Roman Empire), Slide Rule (England, 1620), Pascals Mechanical Calculator (France, 1650), Babbage's Difference Machine (England, 1830), Scheutz' Difference Machine (Sweden, 1850), ENIAC Electronic Numerical Integrator and Computer (USA, 1945), and the Personal Computer PC (USA, 1980), see Fig. 1.5, Fig. 1.6, Fig. 1.7 and Fig. 1.8. The Difference Machines could solve simple differential equations and were used to compute tables of elementary functions such as the logarithm. ENIAC was one of the first modern computers (electronic and programmable), consisted of 18.000 vacuum tubes filling a room of 50×100 square feet with a weight of 30 tons and energy consuming of 200 kilowatts, and was used to solve the differential equations of ballistic firing tables as an important part of the Allied World War II effort. A modern laptop at a cost of \$2000 with a processor speed of 2 GHz and internal mem-

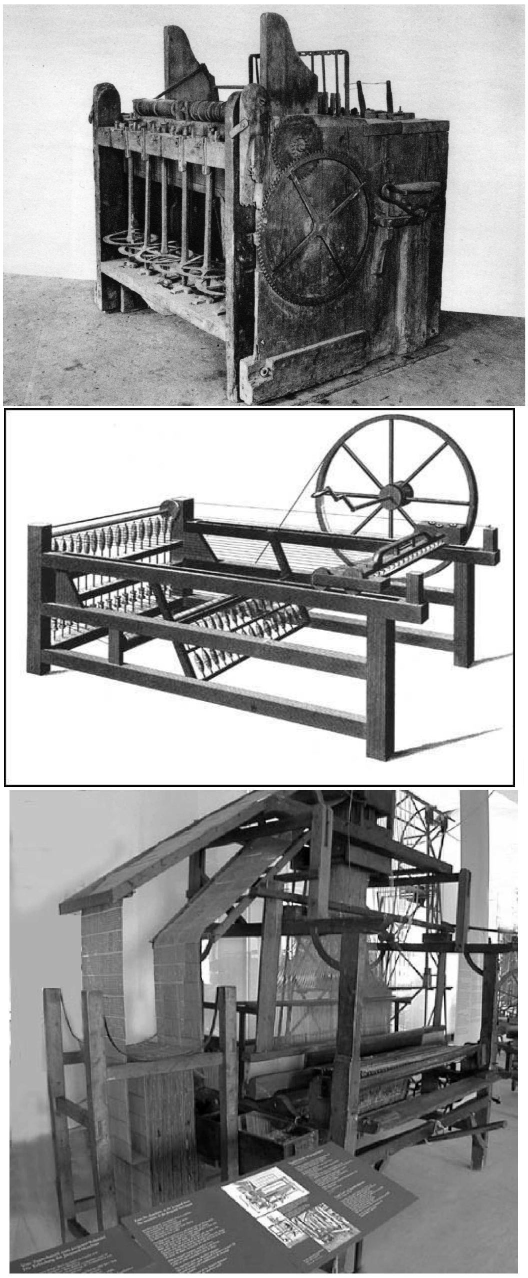

Fig. 1.2. Christoffer Polhem's machine for clock gears (1700), Spinning Jenny (1764) and Jaquard's programmable loom (1801)

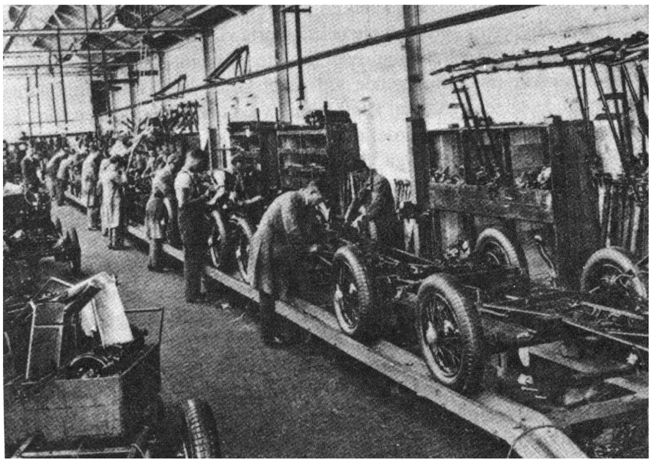

Fig. 1.3. Ford assembly line (1913)

ory of 512 Mb has the computational power of hundreds of thousands of ENIACs.

Automatization (or automation) is based on frequent repetition of a certain *algorithm* or scheme with new data at each repetition. The algorithm may consist of a sequence of relatively simple steps together creating a more complicated process. In automatized manufacturing, as in the production line of a car factory, physical material is modified following a strict repetitive scheme, and in automatized computation, the 1s and 0s of the microprocessor are modified billions of times each second following the computer program. Similarly, a *genetic code* of an organism may be seen as an algorithm that generates a living organism when realized in interplay with the environment. Realizing a genetic code many times (with small variations) generates populations of organisms. Mass-production is the key to increased complexity following the patterns of nature: elementary particle → atom → molecule and molecule → cell → organism → population, or the patterns of our society: individual → group → society or computer → computer network → global net.

1.3 The Role of Mathematics

Mathematics may be viewed as the language of computation and thus lies at the heart of the modern information society. Mathematics is also the language of science and thus lies at the heart of the industrial society that grew out of the *scientific revolution* in the 17th century that began when Leibniz and Newton created *Calculus*. Using Calculus, basic laws of mechanics and physics, such as Newton's law, could be formulated as *mathematical mod-*

Fig. 1.4. Computing device of the Inca Culture

els in the form of *differential equations*. Using the models, real phenomena could be *simulated* and controlled (more or less) and industrial processes could be created.

The mass consumption of both material and immaterial goods, considered to be a corner-stone of our modern democratic society, is made possible through automatization of production and computation. Therefore, mathematics forms a fundamental part of the technical basis of the modern society revolving around automatized production of material goods and automatized computation of information.

The vision of virtual reality based on automatized computation was formulated by Leibniz already in the 17th century and was developed further by Babbage with his Analytical Engine in the 1830s. This vision is finally being realized in the modern computer age in a synthesis of Body & Soul of Mathematics.

We now give some examples of the use of mathematics today that are connected to different forms of automatized computation.

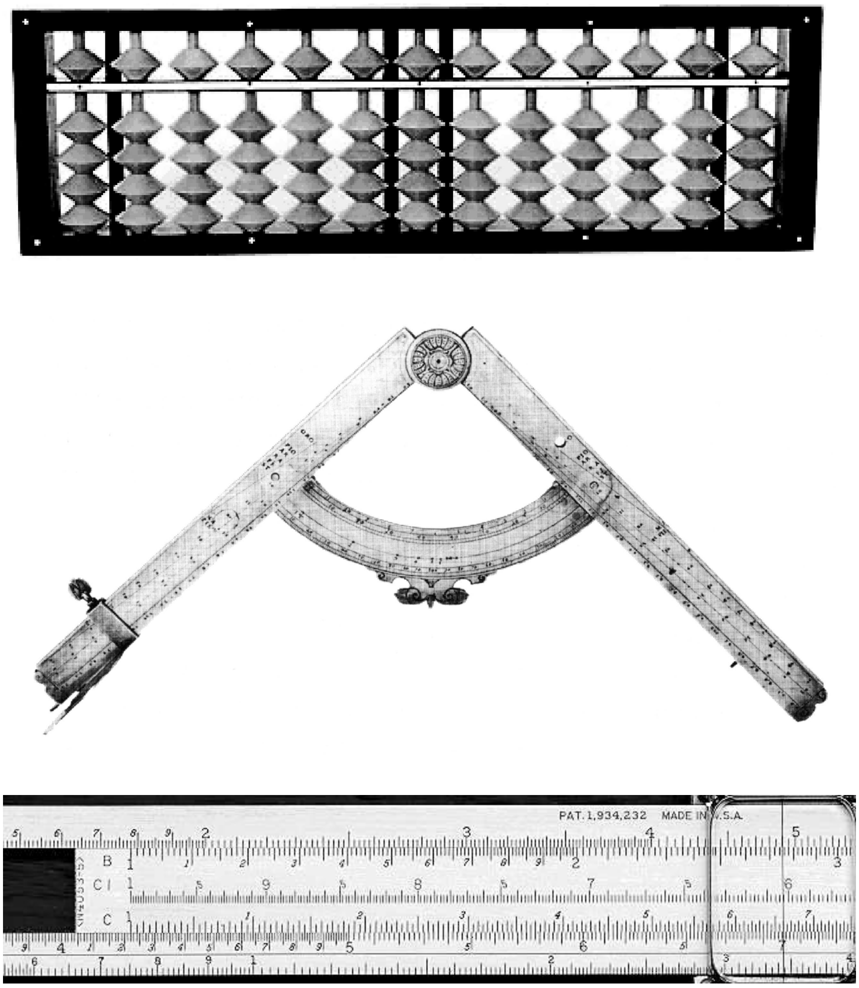

Fig. 1.5. Classical computational tools: Abacus (300 B.C.-), Galileo's Compass (1597) and Slide Rule (1620-)

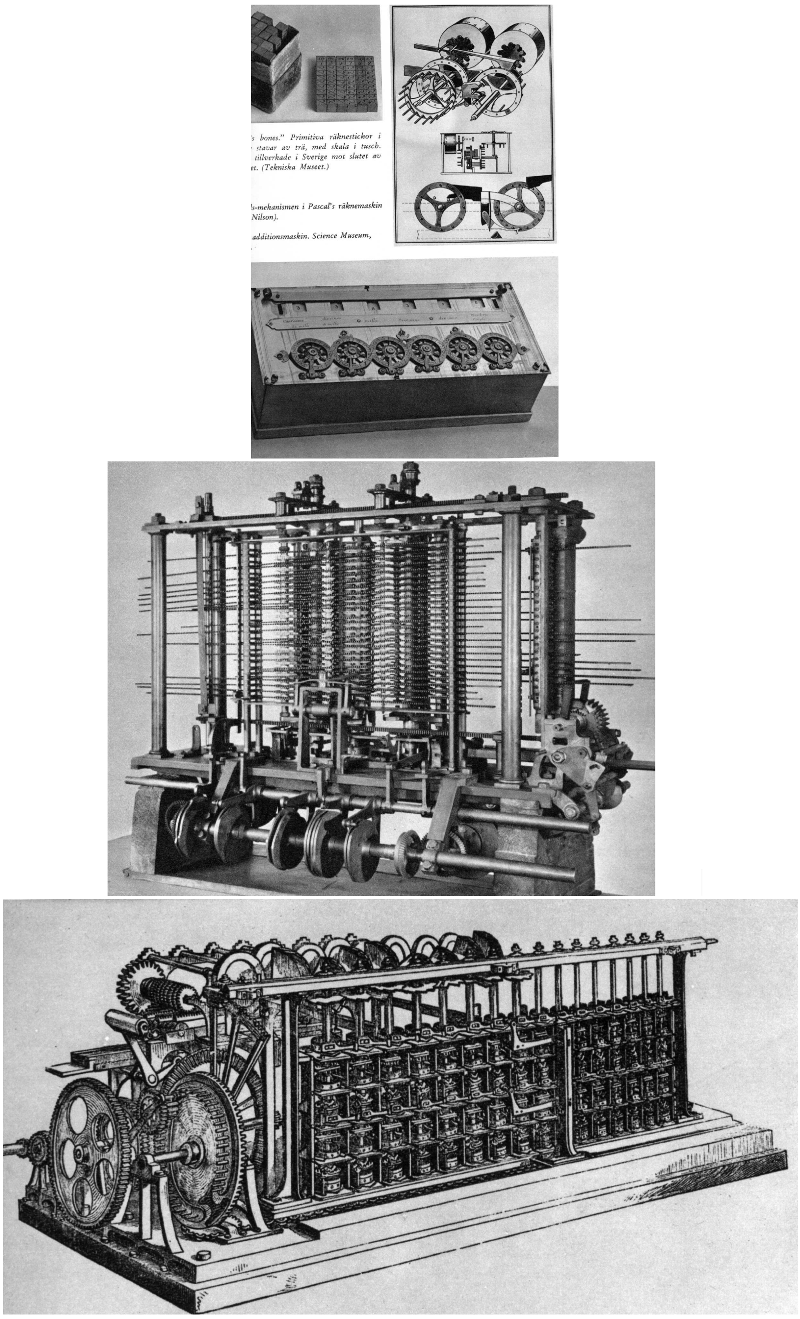

Fig. 1.6. Napier's Bones (1617), Pascals Calculator (1630), Babbage's Difference Machine (1830) and Scheutz' Swedish Difference Machine (1850)

Fig. 1.7. Odhner's mechanical calculator made in Göteborg, Sweden, 1919–1950

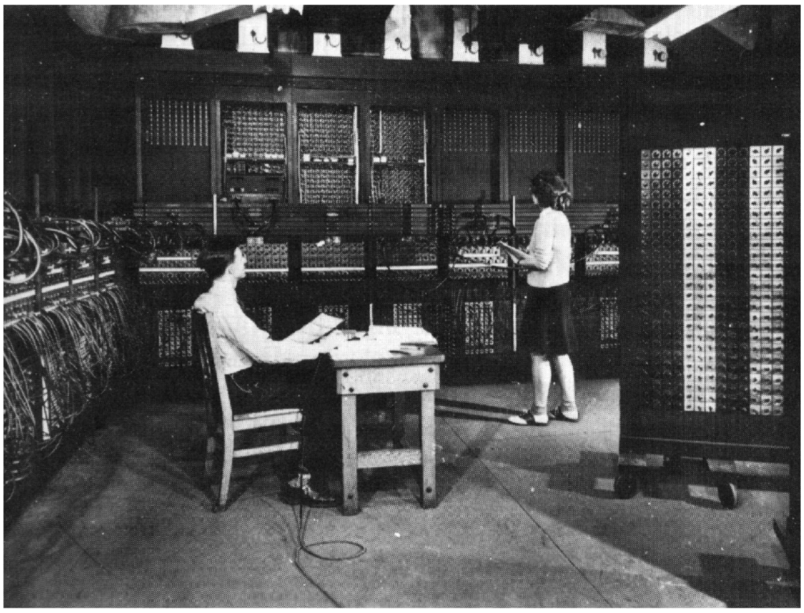

Fig. 1.8. ENIAC Electronic Numerical Integrator and Calculator (1945)

1.4 Design and Production of Cars

In the car industry, a model of a component or complete car can be made using Computer Aided Design CAD. The CAD-model describes the geometry of the car through mathematical expressions and the model can be displayed on the computer screen. The performance of the component can then be tested in computer simulations, where differential equations are solved through massive computation, and the CAD-model is used as input of geometrical data. Further, the CAD data can be used in automatized production. The new technique is revolutionizing the whole industrial process from design to production.

1.5 Navigation: From Stars to GPS

A primary force behind the development of geometry and mathematics since the Babylonians has been the need to navigate using information from the positions of the planets, stars, the Moon and the Sun. With a clock and a sextant and mathematical tables, the sea-farer of the 18th century could determine his position more or less accurately. But the results depended strongly on the precision of clocks and observations and it was easy for large errors to creep in. Historically, navigation has not been an easy job.

During the last decade, the classical methods of navigation have been replaced by GPS, the Global Positioning System. With a GPS navigator in hand, which we can buy for a couple of hundred dollars, we get our coordinates (latitude and longitude) with a precision of 50 meters at the press of a button. GPS is based on a simple mathematical principle known already to the Greeks: if we know our distance to three point is space with known coordinates then we can compute our position. The GPS uses this principle by measuring its distance to three satellites with known positions, and then computes its own coordinates. To use this technique, we need to deploy satellites, keep track of them in space and time, and measure relevant distances, which became possible only in the last decades. Of course, computers are used to keep track of the satellites, and the microprocessor of a hand-held GPS measures distances and computes the current coordinates.

The GPS has opened the door to mass consumption in navigation, which was before the privilege of only a few.

1.6 Medical Tomography

The computer tomograph creates a pictures of the inside of a human body by solving a certain integral equation by massive computation, with data

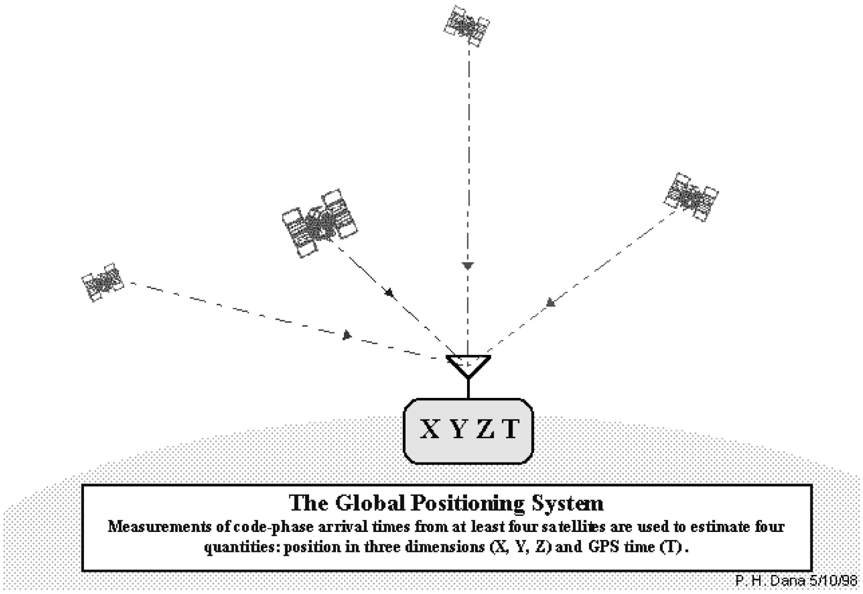

Fig. 1.9. GPS-system with 4 satellites

coming from measuring the attenuation of very weak X-rays sent through the body from different directions. This technique offers mass consumption of medical imaging, which is radically changing medical research and practice.

1.7 Molecular Dynamics and Medical Drug Design

The classic way in which new drugs are discovered is an expensive and time-consuming process. First, a physical search is conducted for new organic chemical compounds, for example among the rain forests in South America. Once a new organic molecule is discovered, drug and chemical companies license the molecule for use in a broad laboratory investigation to see if the compound is useful. This search is conducted by expert organic chemists who build up a vast experience with how compounds can interact and which kind of interactions are likely to prove useful for the purpose of controlling a disease or fixing a physical condition. Such experience is needed to reduce the number of laboratory trials that are conducted, otherwise the vast range of possibilities is overwhelming.

The use of computers in the search for new drugs is rapidly increasing. One use is to makeup new compounds so as to reduce the need to make expensive searches in exotic locations like southern rain forests. As part of this search, the computer can also help classify possible configurations of

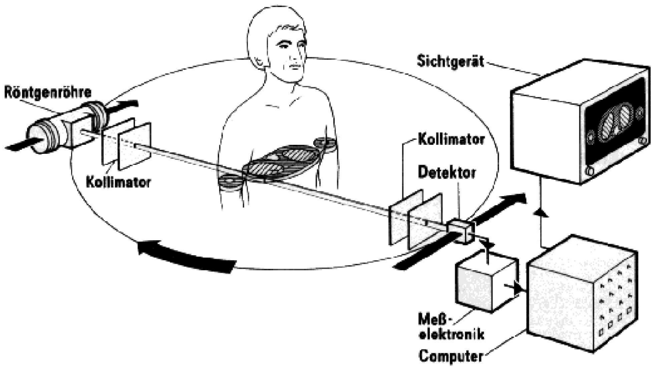

Fig. 1.10. Medical tomograph

molecules and provide likely ranges of interactions, thus greatly reducing the amount of laboratory testing time that is needed.

1.8 Weather Prediction and Global Warming

Weather predictions are based on solving differential equations that describe the evolution of the atmosphere using a super computer. Reasonably reliable predictions of daily weather are routinely done for periods of a few days. For longer periods. the reliability of the simulation decreases rapidly, and with present day computers daily weather predictions for a period of two weeks are impossible.

However, forecasts over months of averages of temperature and rainfall are possible with present day computer power and are routinely performed.

Long-time simulations over periods of 20–50 years of yearly temperature-averages are done today to predict a possible *global warming* due to the use of fossil energy. The reliability of these simulations are debated.

1.9 Economy: Stocks and Options

The Black-Scholes model for pricing options has created a new market of so called derivative trading as a complement to the stock market. To correctly price options is a mathematically complicated and computationally intensive task, and a stock broker with first class software for this purpose (which responds in a few seconds), has a clear trading advantage.

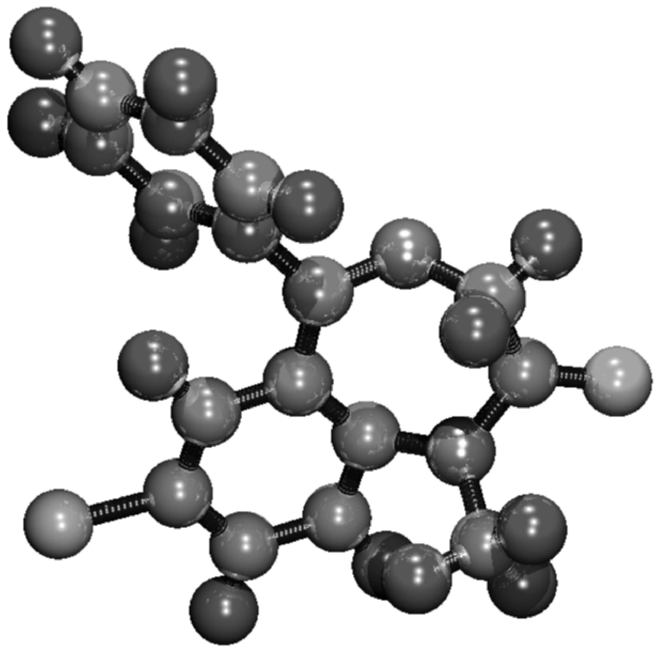

Fig. 1.11. The Valium molecule

1.10 Languages

Mathematics is a *language*. There are many different languages. Our mother tongue, whatever it happens to be, English, Swedish, Greek, et cetera, is our most important language, which a child masters quite well at the age of three. To learn to write in our native language takes longer time and more effort and occupies a large part of the early school years. To learn to speak and write a foreign language is an important part of secondary education.

Language is used for *communication* with other people for purposes of cooperation, exchange of ideas or control. Communication is becoming increasingly important in our society as the modern means of communication develop.

Using a language we may create *models* of phenomena of interest, and by using models, phenomena may be studied for purposes of *understanding* or *prediction*. Models may be used for *analysis* focussed on a close examination of individual parts of the model and for *synthesis* aimed at understanding the interplay of the parts that is understanding the model as a whole. A *novel* is like a model of the real world expressed in a written language like English. In a novel the characters of people in the novel may be analyzed and the interaction between people may be displayed and studied.

The ants in a group of ants or bees in a bees hive also have a language for communication. In fact in modern biology, the interaction between cells or proteins in a cell is often described in terms of entities "talking to each other".

It appears that we as human beings use our language when we *think*. We then seem to use the language as a model in our head, where we try various possibilities in *simulations* of the real world: "If that happens, then I'll do this, and if instead that happens, then I will do so and so...". Planning our day and setting up our calender is also some type of modeling or simulation of events to come. Simulations by using our language thus seems to go on in our heads all the time.

There are also other languages like the language of musical notation with its notes, bars, scores, et cetera. A musical score is like a model of the real music. For a trained composer, the model of the written score can be very close to the real music. For amateurs, the musical score may say very little, because the score is like a foreign language which is not understood.

1.11 Mathematics as the Language of Science

Mathematics has been described as the language of science and technology including mechanics, astronomy, physics, chemistry, and topics like fluid and solid mechanics, electromagnetics et cetera. The language of mathematics is used to deal with *geometrical* concepts like *position* and *form* and *mechanical* concepts like *velocity*, *force* and *field*. More generally, mathematics serves as a language in any area that includes *quantitative* aspects described in terms of *numbers*, such as economy, accounting, statistics et cetera. Mathematics serves as the basis for the modern means of electronic *communication* where information is coded as sequences of 0's and 1's and is transferred, manipulated or stored.

The words of the language of mathematics often are taken from our usual language, like *points*, *lines*, *circles*, *velocity*, *functions*, *relations*, *transformations*, *sequences*, *equality*, *inequality* et cetera.

A mathematical word, term or concept is supposed to have a specific meaning defined using other words and concepts that are already defined. This is the same principle as is used in a Thesaurus, where relatively complicated words are described in terms of simpler words. To start the definition process, certain fundamental concepts or words are used, which cannot be defined in terms of already defined concepts. Basic relations between the fundamental concepts may be described in certain *axioms*. Fundamental concepts of Euclidean geometry are *point* and *line*, and a basic Euclidean axiom states that through each pair of distinct points there is a unique line passing. A *theorem* is a statement derived from the axioms or other theorems by using logical reasoning following certain rules of logic. The derivation is called a *proof* of the theorem.

1.12 The Basic Areas of Mathematics

The basic areas of mathematics are

- Geometry
- Algebra
- Analysis.

Geometry concerns objects like *lines*, *triangles*, *circles*. Algebra and Analysis is based on *numbers* and *functions*. The basic areas of mathematics education in engineering or science education are

- Calculus
- Linear Algebra.

Calculus is a branch of analysis and concerns properties of functions such as *continuity*, and operations on functions such as *differentiation* and *integration*. Calculus connects to Linear Algebra in the study of *linear functions* or linear transformations and to *analytical geometry*, which describes geometry in terms of numbers. The basic concepts of Calculus are

- function
- derivative
- integral.

Linear Algebra combines Geometry and Algebra and connects to Analytical Geometry. The basic concepts of Linear Algebra are

- vector
- vector space
- projection, orthogonality
- linear transformation.

This book teaches the basics of Calculus and Linear Algebra, which are the areas of mathematics underlying most applications.

1.13 What Is Science?

The theoretical kernel of *natural science* may be viewed as having two components

- formulating equations (modeling),
- solving equations (computation).

Together, these form the essence of *mathematical modeling* and *computational mathematical modeling.* The first really great triumph of science and mathematical modeling is Newton's model of our planetary system as a set of differential equations expressing Newton's law connecting force, through the inverse square law, and acceleration. An *algorithm* may be seen as a strategy or constructive method to solve a given equation via computation. By applying the algorithm and computing, it is possible to simulate real phenomena and make predictions.

Traditional techniques of computing were based on symbolic or numerical computation with pen and paper, tables, slide ruler and mechanical calculator. Automatized computation with computers is now opening new possibilities of simulation of real phenomena according to Natures own principle of massive repetition of simple operations, and the areas of applications are quickly growing in science, technology, medicine and economics.

Mathematics is basic for both steps (i) formulating and (ii) solving equation. Mathematics is used as a language to formulate equations and as a set of tools to solve equations.

Fame in science can be reached by formulating or solving equations. The success is usually manifested by connecting the name of the inventor to the equation or solution method. Examples are legio: Newton's method, Euler's equations, Lagrange's equations, Poisson's equation, Laplace's equation, Navier's equation, Navier-Stokes' equations, Boussinesq's equation, Einstein's equation, Schrödinger's equation, Black-Scholes formula..., most of which we will meet below.

1.14 What Is Conscience?

The activity of the brain is believed to consist of electrical/chemical signals/waves connecting billions of synapses in some kind of large scale computation. The question of the nature of the *conscience* of human beings has played a central role in the development of human culture since the early Greek civilization, and today computer scientists seek to capture its evasive nature in various forms of Artificial Intelligence AI. The idea of a division of the activity of the brain into a (small) *conscious* "rational" part and a (large) *unconscious* "irrational" part, is widely accepted since the days of Freud. The rational part has the role of "analysis" and "control" towards some "purpose" and thus has features of Soul, while the bulk of the "computation" is Body in the sense that it is "just" electrical/chemical waves. We meet the same aspects in numerical optimization, with the optimization algorithm itself playing the role of Soul directing the computational effort towards the goal, and the underlying computation is Body.

We have been brought up with the idea that the conscious is in control of the mental "computation", but we know that this is often not the case. In fact, we seem to have developed strong skills in various kinds of after-rationalization: whatever happens, unless it is an "accident" or something "unexpected", we see it as resulting from a rational plan of ours made up in advance, thus turning a posteriori observations into a priori predictions.

1.15 How to Come to Grips with the Difficulties of Understanding the Material of this Book and Eventually Viewing it as a Good Friend

We conclude this introductory chapter with some suggestions intended to help the reader through the most demanding first reading of the book and reach a state of mind viewing the book as a good helpful friend, rather than the opposite. From our experience of teaching the material of this book, we know that it may evoke quite a bit of frustration and negative feelings, which is not very productive.

Mathematics Is Difficult: Choose Your Own Level of Ambition

First, we have to admit that mathematics is a difficult subject, and we see no way around this fact.Secondly, one should realize that it is perfectly possible to live a happy life with a career in both academics and industry with only elementary knowledge of mathematics. There are many examples including Nobel Prize Winners. This means that it is advisable to set a level of ambition in mathematics studies which is realistic and fits the interest profile of the individual student. Many students of engineering have other prime interests than mathematics, but there are also students who really like mathematics and theoretical engineering subjects using mathematics. The span of mathematical interest thus may be expected to be quite wide in a group of students following a course based on this book, and it seems reasonable that this would be reflected in the choice of level of ambition.

Advanced Material: Keep an Open Mind and Be Confident

The book contains quite a bit of material which is "advanced" and not usually met in undergraduate mathematics, and which one may bypass and still be completely happy. It is probably better to be really familiar with and understand a smaller set of mathematical tools and have the ability to meet new challenges with some self-confidence, than repeatedly failing to digest too large portions. Mathematics is so rich, that even a life of fully-time study can only cover a very small part. The most important ability must be to meet new material with an open mind and some confidence!

Some Parts of Mathematics Are Easy

On the other hand, there are many aspects of mathematics which are not so difficult, or even "simple", once they have been properly understood. Thus, the book contains both difficult and simple material, and the first impression from the student may give overwhelming weight to the former. To help out we have collected the most essential nontrivial facts in short summaries in the form of *Calculus Tool Bag I and II*, *Linear Algebra Tool Bag*, *Differential Equations Tool Bag*, *Applications Tool Bag*, *Fourier Analysis Tool Bag* and *Analytic Functions Tool Bag.* The reader will find the tool bags surprisingly short: just a couple pages, altogether say 15–20 pages. If properly understood, this material carries a long way and is "all" one needs to remember from the math studies for further studies and professional activities in other areas. Since the book contains about 1200 pages it means 50–100 pages of book text for each one page of summary. This means that the book gives more than the absolute minimum of information and has the ambition to give the mathematical concepts a perspective concerning both history and applicability today. So we hope the student does not get turned off by the quite a massive number of words, by remembering that after all 15–20 pages captures the essential facts. During a period of study of say one year and a half of math studies, this effectively means about one third of a page each week!

Increased/Decreased Importance of Mathematics

The book reflects both the increased importance of mathematics in the information society of today, and the decreased importance of much of the analytical mathematics filling the traditional curriculum. The student thus should be happy to know that many of the traditional formulas are no longer such a must, and that a proper understanding of relatively few basic mathematical facts can help a lot in coping with modern life and science.

Which Chapters Can I Skip in a First Reading?

We indicate by * certain chapters directed to applications, which one may by-pass in a first reading without loosing the main thread of the presentation, and return to at a later stage if desired.

Chapter 1 Problems

1.1. Find out which Nobel Prize Winners got the prize for formulating or solving equations.

1.2. Reflect about the nature of “thinking” and “computing”.

1.3. Find out more about the topics mentioned in the text.

1.4. (a) Do you like mathematics or hate mathematics, or something in between? Explain your standpoint. (b) Specify what you would like to get out of your studies of mathematics.

1.5. Present some basic aspects of science.

Fig. 1.12. Left person: “Isn’t it remarkable that one can compute the distance to stars like Cassiopeja, Aldebaran and Sirius?”. Right person: “I find it even more remarkable that one may know their names!” (Assar by Ulf Lundquist)