GAUSSIAN ELIMINATION

COMPUTER SESSION A8

Background

We have previously examined properties of linear systems. We will now focus on a specific solution algorithm for linear systems: Gaussian Elimination. It is the standard algorithm for solving a linear system by hand and is also used widely in practice.

Before today's computer session, make sure that you understand and can answer the following questions.

Question 1

What is a linear system?

Problems

Problem 1 - Gaussian Elimination.

(1) Consider the 3×3 matrix `A = [ 0 -1 2; 1 -1 0; 3 1 -1]` and the (column) vector `b = [ 7; 3; 2]`, and solve $Ax = b$ using Gaussian elimination (by hand computation). Check your solution by computing `A * x`, to see that you get the result b, and compare to Matlab's A b.

Construct the matrix `S = [A, b]` representing the linear system. The function `rref(B)` (where B is a matrix) computes the resulting matrix of Gaussian elimination. If you have trouble understanding what it does, try `rrefmovie(S)` on your linear system S.

Note that there are two variants of Gaussian elimination:

The first variant, described in the AMBS book, stops when the matrix A in the system S has been transformed into a *triangular* matrix. It is then possible to simply substitute the values in starting from the bottom to get the solution x.

The second variant, which `rref(B)` uses, continues and transforms the matrix A in the system S into a *diagonal* matrix with ones on the diagonal (this matrix is called the *identity* matrix). The solution x is then simply the column vector to the right in S.

(2) Solve (by hand) the equation system $Ax = b$ where `A = [ 1 2 3; -2 -4 3; 3 1 0]` and `b = [ 7; 13; -6]`. Check your answer with `rref([A b])`. Compare with the operation `A \ b`.

(3) Solve using matlab $Ax = b$ with A the same as in 2, `b = e1 = [ 1; 0; 0]`, `b = e2 = [ 0; 1; 0]` and `b = e3 = [ 0; 0; 1]`, respectively. Denote the solutions by x1, x2 and x3 , and put the three solutions together to form the matrix `C = [x1 x2 x3]`, and verify that $AC = I = (e_1, e_2, e_3)$. Recall that the matrix C is called the inverse of A , and is denoted by A^{-1}, or `inv(A)` in Matlab syntax, that is `C == inv(A)`.

Now compute `inv(A) * b` , with b as in 2 , and note that this gives the solution to $Ax = b$. Why?

(4) Compute the determinant `det(A)` of A (check by hand computation), that is the volume spanned by the three column vectors of A, (see section 21.11 of AMBS). Note that this number determines if the linear system of equations $Ax = b$ has a unique solution or not for all b. Is this the case for the A under consideration ?

(5) Consider now a singular matrix, for example `A = [ 2 1 -1; 1 3 0; 0 -5 -1]`, where the second column is a linear combination of the first and third one (can you see how?). Likewise the second row is a combination of the first and third. Now seek to solve $Ax = b$ with `b = [ 1; 2; 3]` using Gaussian elimination. What happens? Compute the determinant of A and try to explain the result to a comrade.

Try solving the same system using the Matlab \ operator. How do you interpret Matlabs answer? Is there an inverse matrix such that $A^{-1}A = I$ (`inv(A) * A == I` in Matlab syntax)?

Problem 2 - Applications of solving linear systems.

(1) Linear Transformations

The matrix $R_z = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ represents the rotation of the angle θ around the z-axis. $y = R_z x$ gives that y will be x rotated θ degrees around the z-axis.

If you have time, visualize a rotation with the graphics system used in the A7 session. Draw both x and $R_z x$ for some vectors x.

(2) Inverting Linear Transformations

The equation $R_z x = b$, where R_z is the rotation matrix given above, x is unknown and b is a known vector, gives that x will be rotated by R_z and produce b. But here we only have b, we don't have x. Here we need to solve the equation $R_z x = b$ to find x. Select a simple angle θ ($\frac{pi}{2}$ for example) and solve $R_z x = b$ using Gaussian elimination, verify with Matlab.

If you have time, visualize this new situation with the graphics system. Draw both x and b and verify visually that x is rotated back from b.

(3) More Linear Transformations

Do the same for other linear transformations from the AMBS book. Which ones are invertible and which ones aren't?

About

This Computer Session is part of the Body and Soul educational program. More information can be found at

`http://www.phi.chalmers.se/bodysoul/`

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