LINEARIZATION, NEWTON'S METHOD

COMPUTER SESSION D3

BACKGROUND

Question 1

PREPARATIONS

The session is divided into two parts. The first part involves experimenting in the Mathematics Laboratory and the second part involves programming with vectors and matrices.

Start Matlab.

If you are working on the computers of the School of Chemical Engineering at Chalmers, then download the file `startmath.m` to your Matlab work directory (if you have not done this already). This file is available on the web page of this session under *Programs and templates*. Then type `startmath` at the Matlab prompt. This command sets the search path to the directories where the Mathematics Laboratory is kept.

If you are working on another computer, then download the file `MathematicsLaboratory.zip` to your Matlab work directory. This file is available on the web page of this session under *Programs and templates*. Unzip the file, it should create a directory guis in your Matlab work directory. At the Matlab prompt, type: `addpath guis`. You are now ready to use the Mathematics Laboratory.

Keep your AMBS book with you and open at the relevant chapters.

PROBLEMS

Problem 1 - Linearization in the Mathematics Laboratory.

(1) Give the command `open('RM+.fig')` to open the Road Map to the Mathematics Laboratory, and press the Linearization button. Alternatively you can start the lab directly from the matlab prompt by the command `open('LIN.fig')`.

(2) For the following problems, verify by hand computation using the definition of the tangent or linearization $h(x)$, of a function $f(x)$ in the point $\hat{x}$:

$$h(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x})$$

Also verify visually using the Linearization Lab in Matlab.

Verify that the tangent to $f(x) = x^3$ for $x = 0.5$ is $h(x) = 0.125 + 0.75(x - 0.5)$. Verify that the tangent to $f(x) = \frac{1}{x^2+1}$ for $x = 1$ is $h(x) = 0.5 - 0.5(x - 1)$. Find the tangent to $f(x) = \frac{1}{x^2+1}$ for $x = -0.5$. Zoom in to make sure you really found the tangent. What is the tangent to $f(x) = \sqrt{x^2 + 1}$ for $x = 1$?

(3) Give the command `open('RM+.fig')` to open the Road Map to the Mathematics Laboratory, and press the Newton R button. Alternatively you can start the lab directly from the matlab prompt by the command `open('NEWTR.fig')`.

(4) The Newton Lab finds solutions to equations $f(x) = 0$ by starting with a point x_i, finding the tangent of $f(x)$ (a linear approximation of $f(x)$), then finds the solution of the tangent equation, which is a simple approximation of the real equation. The solution of the tangent equation is named x_{i+1}, and is then used as the starting value for the next iteration.

In the Newton Lab, you compute the linearization by pressing the button “linearize”, and find the solution of the tangent equation by pressing “iterate”.

Solve the equation $x^2 - 0.1 = 0$ with the Newton Lab.

Find both roots of the equation $x^2 + 2x - 1 = x$.

(5) Let’s say that x_i is the initial guess, or current approximate solution, of the solution to $f(x) = 0$. x_{i+1} is the next approximation, computed by Newton’s method.

For each step of the algorithm, we are finding the solution of the equation:

$$h(x) = 0$$

where $h(x)$ is the tangent of $f(x)$. The solution is then the next approximate solution x_{i+1} of $f(x) = 0$.

If we insert the definition of the tangent for $f(x)$ in the point x_i, then we get:

$$f(x_i) + f'(x_i)(x_{i+1} - x_i) = 0$$

Solve this equation (symbolically) for x_{i+1}, so you have $x_{i+1} =$ ....

You should now have Newton's method in fixed point form. We can now proceed to implement the algorithm, just as we did with the general fixed point algorithm.

Problem 2 - Implementing Newton's Method. Since Newton's method is a fixed point algorithm, we can simply use our previous implementation of the fixed point algorithm. Use the version in *Programs and templates* or use the version you wrote yourself.

We previously implemented the fixed point algorithm with $x = g(x)$ and $g(x) = x - af(x)$. So we had the iteration:

$$x_{i+1} = x_i - af(x_i)$$

where we had to choose a depending on which $f(x)$ we put in.

Newton's method says that we should choose $a = \frac{1}{f'(x_i)}$.

So we get:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

We compute the derivative $f'(x)$ by numerical differentiation.

(1) Numerical Differentiation

Write a function `derivative(f, a)` which takes a function $f(x)$ and a number a as input arguments and returns the derivative $f'(a)$. Use the formula:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

where h is a small number (an analysis of the right choice of h exists in the AMBS book). $h = 1e - 8$ works well for most cases.

To be able to give a function as an input argument, we can give the *name* of the function put in quotes. For example, to get the derivative of $sin(0)$, we want to use our function as `derivative('sin', 0)`.

In the implementation of the function, we can evaluate a function by its name by using the function `feval(f, x)`. For example, we have that:

```
>> feval('sqrt', 2)

ans =

    1.4142
```

(2) Newton's Method

Write a function `fnewton(f, guess, tol)` which implements Newton's method based on the previous `ffixedpoint()` and using the `derivative(f, a)` function.

(3) Testing your implementation

Solve the same equations as you did with the Newton Lab and check that you get the same answers.

Your function should behave like this:

```
>> x = fnewton('f', 1, 1e-6)

x =

    1.4142
```

Where `f.m` contains:

```
function y = f(x)
% f(x)
% Returns x^2 - 2, representing the equation x^2 = 2

  y = x^2 - 2;
```

SOLUTIONS

Make sure that you really try to solve each problem before looking at the solutions. Have you really tried to solve the problem or should you try again before looking at the solution?

The solutions are available on the web page of this session under *Solutions to problems*.

ABOUT

This Computer Session is part of the Body and Soul educational program. More information can be found at

`http://www.phi.chalmers.se/bodysoul/`

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