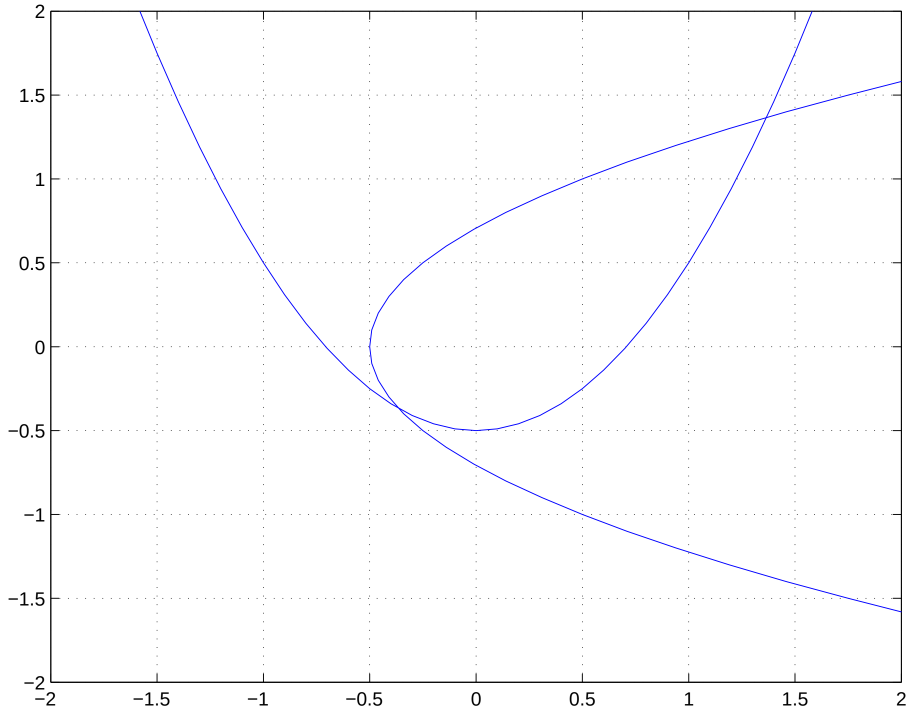
NEWTON'S METHOD FOR SYSTEMS

COMPUTER SESSION E9



$$x_1^2 - x_2 = 0.5$$
$$-x_1 + x_2^2 = 0.5$$

Background

Question 1

PREPARATIONS

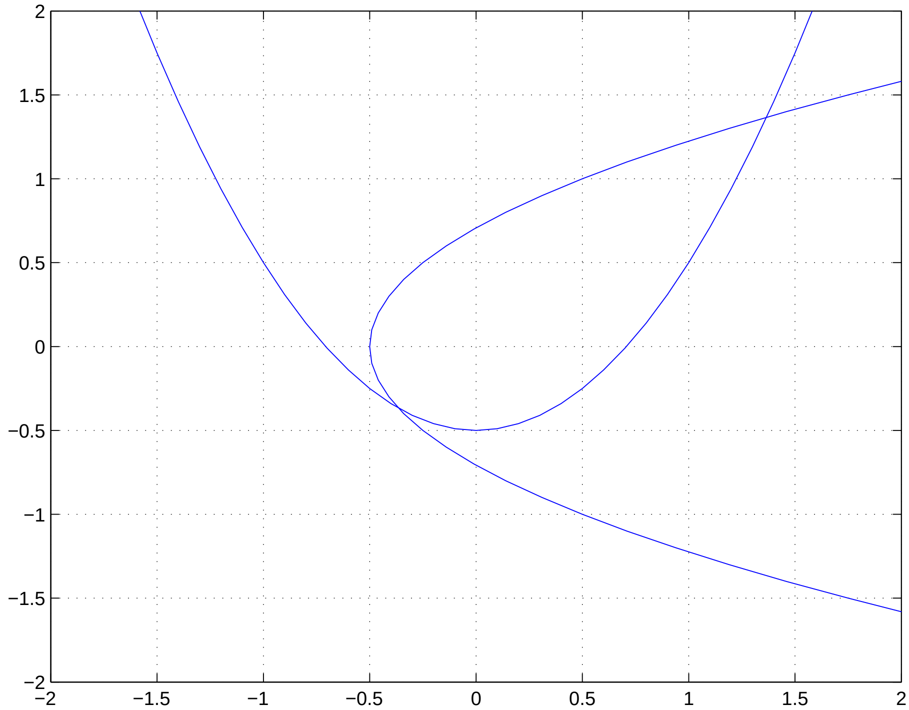
PROBLEMS

Problem 1 - f(x) = 0 for systems. We have previously defined an equation as $f(x) = 0$. For example, if we have the equation $x^2 = 2$, then we would move everything to the left hand side and get $x^2 - 2 = 0$, and thus $f(x) = x^2 - 2$.

We now start with a system of equations. We have several equations and several variables. In this session we will only consider systems of two equations and two variables. For example, we could have the system:

$$\begin{aligned} x_1^2 - x_2 &= 0.5 \\ -x_1 + x_2^2 &= 0.5 \end{aligned}$$

Graphically, each equation can be seen as a curve, and the solutions of the system are then the intersections of the curves (remember how a 2×2 linear system could be seen as two lines):



To put this system in $f(x) = 0$ form, we proceed just like before, by moving everything to the left hand side:

$$\begin{aligned} x_1^2 - x_2 - 0.5 &= 0 \\ -x_1 + x_2^2 - 0.5 &= 0 \end{aligned}$$

We then define the left hand sides as two functions, $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$. I.e. functions of two variables which return one value:

$$\begin{aligned} f_1(x_1, x_2) &= x_1^2 - x_2 - 0.5 \\ f_2(x_1, x_2) &= -x_1 + x_2^2 - 0.5 \end{aligned}$$

We can then define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is vector-valued and has f_1 and f_2 as components:

$$f(x_1, x_2) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

We can also see the input arguments x_1 and x_2 as elements of a vector:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then we can write:

$$f(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

and we have the form:

$$f(x) = 0$$

(note that 0 is the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in this context).

(1) Implementing the function $f(x)$

Implement the above system of equations as the function `f(x)` which takes a 2-element column vector as input argument and returns a two-element column vector as output.

Note: write:

```
function y = f(x)

  y = zeros(2, 1);
  ...
```

to force y to be a column vector.

Your function should behave like this:

```
>> x = f([1; 1])


ans =

   -0.5000
   -0.5000
```

Test your function for some input arguments x and see if you can come close to a solution.

Problem 2 - Implementing Newton's Method. Fixed point iteration for a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ works just the same as for a function $f : \mathbb{R} \to \mathbb{R}$.

We start with $f(x) = 0$, and we want to put this equation in the form $x = g(x)$.

$$\begin{aligned}
f(x) = 0 \Rightarrow \text{(Multiply with a matrix } A\text{)} \\
Af(x) = 0 \Rightarrow \text{(add } x\text{)} \\
x + Af(x) = x \\
\Rightarrow \\
g(x) = x + Af(x) \\
\Rightarrow \\
x = g(x)
\end{aligned}$$

We could select matrices A depending on the function $f(x)$ and successfully use simple fixed point iteration. However, we will go directly to Newton's method (which is a systematic way of finding the best matrix A).

Just as for the case of only one equation, we linearize $f(x)$ to get the tangents (the tangents are now planes, since we have two variables, and we get two tangents, since $f(x)$ has two components) $t(x)$ in the point $\hat{x}$:

$$t(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x})$$

Just as before, we find the solution to $t(x) = 0$ instead and hope that it is a good approximation of the solution to $f(x) = 0$.

If we denote $\hat{x}$ as our starting guess or current approximate solution x_i and x as the next approximate solution x_{i+1}, we get:

$$f(x_i) + f'(x_i)(x_{i+1} - x_i) = 0$$

and just as before we solve for x_{i+1}:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

This time however, $f'(x_i)$ is a matrix (sometimes called Jacobi or Jacobian matrix) composed of all the combinations of derivatives with respect to the two variables and two components:

$$f'(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

Instead of explicitly computing the inverse of $f'(x_i)$, it is more efficient (it takes less operations) to solve for $x_i - x_{i+1}$ instead to get:

$$f'(x_i)(x_{i+1} - x_i) = -f(x_i)$$

If we denote $x_{i+1} - x_i$ as h, $f'(x_i)$ as J and $-f(x_i)$ as r we get a familiar linear system:

$$Jh = r$$

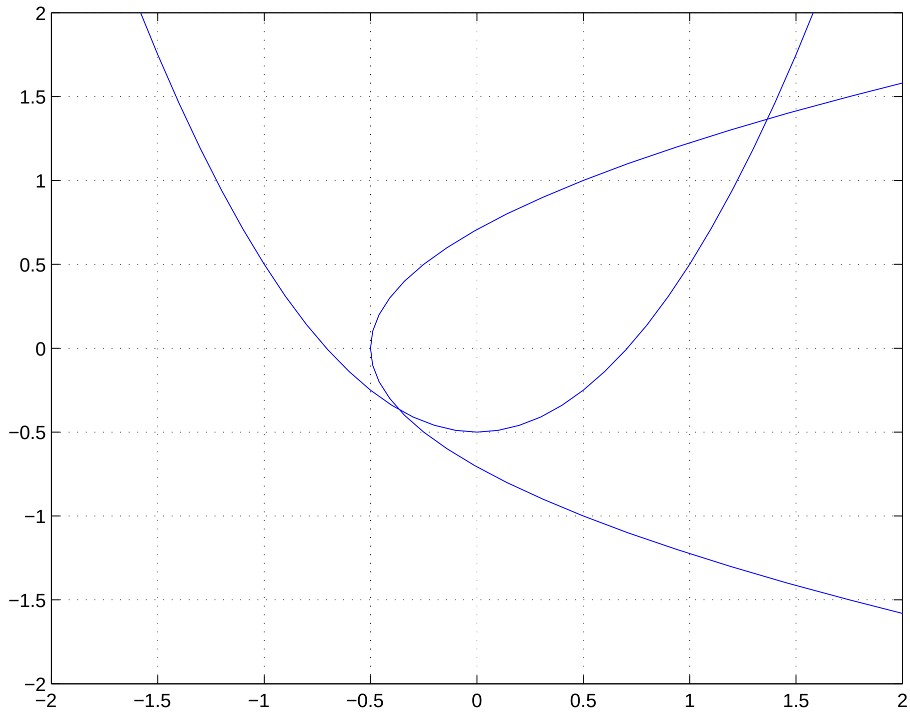
where h is the unknown. We can then compute $x_{i+1} = h + x_i$.

(1) Newton's Method
Write a function `fnewtonsystem(f, guess, tol)` which implements Newton's method based on the previous `fnewton()` and using the `jacobi(f, a)` function to compute the derivative. Both functions are in http://www.phi.chalmers.se/bodysoul/sessions/e9/programs/.
Note: you should now check the norm of your residual instead of only the absolute value since the residual is now a vector. The same applies if you want to look at the difference between the new and old x.

(2) Testing your implementation
Solve the previously mentioned equation system:

$$\begin{aligned} x_1^2 - x_2 - 0.5 &= 0 \\ -x_1 + x_2^2 - 0.5 &= 0 \end{aligned}$$

and verify visually and by hand computation.



Your function should behave like this:

```
>> x = fnewtonsystem('f', [1; 1], 1e-6)

x =

    1.3660
    1.3660
```

Where `f.m` is your implementation of $f(x)$ representing the equation system.

Solutions

Make sure that you really try to solve each problem before looking at the solutions. Have you really tried to solve the problem or should you try again before looking at the solution?

The solutions are available on the web page of this session under *Solutions to problems*.

About

This Computer Session is part of the Body and Soul educational program. More information can be found at

`http://www.phi.chalmers.se/bodysoul/`

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