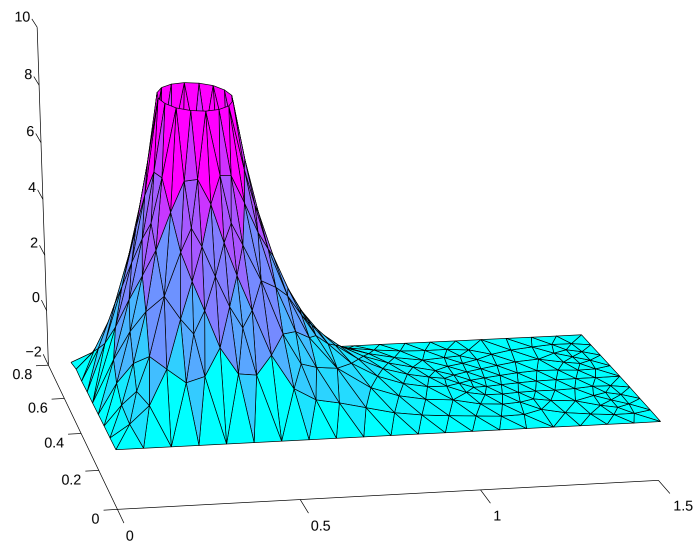
Back to Poisson

Computer Session E1



Background

The variational formulation of Poisson's equation

$$\begin{aligned} -\nabla \cdot (a\nabla u) &= f \quad \text{in } \Omega, \\ -a\partial_n u &= \gamma(u - g_D) + g_N \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

is given by

$$\int_\Omega a\nabla u \cdot \nabla v \, dx + \int_\Gamma \gamma u v \, ds = \int_\Omega f v \, dx + \int_\Gamma (\gamma g_D - g_N) v \, ds \quad \forall v. \tag{2}$$

Using this general form, we can specify both Dirichlet boundary conditions (values of u on the boundary) and Neumann boundary conditions (values of $\partial_n u$) on the boundary).

Before today's computer session, make sure that you understand and can answer the following questions.

Question 1 Derive the variational formulation (2) from Poisson's equation (1).

Question 2 How do you specify Dirichlet and Neumann boundary conditions using the general boundary condition in (1)?

Question 3 Verify that $u(x) = \sin(\pi x_1)\sin(2\pi x_2)$ is a solution of Poisson's equation on the unit square $\Omega = (0,1) \times (0,1)$ with right-hand side $f(x) = 5\pi^2 \sin(\pi x_1)\sin(\pi x_2)$. What are the boundary conditions?

Preparations

Create a new directory called `e1` and the three subdirectories `problem1`, `problem2`, and `problem3`. Then download the following files to each of the subdirectories:

- `AssembleMatrix.m`,
- `AssembleVector.m`,
- `Poisson.m`,
- the files in the directory `mesh`.

These files are available on the web page of this session under *Programs and templates*.

Problems

Problem 1

Preparation

Go to the directory `problem1`. Create a file called `PoissonSolver.m` and write your program in this file. You also have to edit the file `Poisson.m`, where you specify the variational formulation.

Problem

Solve Poisson's equation on the unit square (`square.m`) with homogeneous Dirichlet boundary conditions, $a = 1$, and the right-hand side given by

$$f(x) = 5\pi^2 \sin(\pi x_1) \sin(2\pi x_2). \tag{3}$$

Compare your computed solution $U(x)$ with the exact solution $u(x)$. How large is the error in the maximum norm? The maximum norm of the error is given by

$$\|e\|_\infty = \|U - u\|_\infty = \max_{x\in\Omega} |U(x) - u(x)|. \tag{4}$$

How large is the error when you refine the mesh (`square_refined.m`)?

Check your answer: The error in the maximum norm should be 0.0216 for the first mesh and 0.0055 for the refined mesh. Notice that the error is decreased by a factor $4 = 2^2$ when we decrease the mesh size with a factor 2.

Problem 2

Preparation

Go to the directory `problem2`. Create a file called `PoissonSolver.m` and write your program in this file. You also have to edit the file `Poisson.m`, where you specify the the variational formulation.

Problem

Solve Poisson's equation on the geometry given in Figure 1 with $f = 0$. The mesh for this geometry is provided in the file `cylinder.m`. You could also use the refined mesh in the file `cylinder_refined.m`.

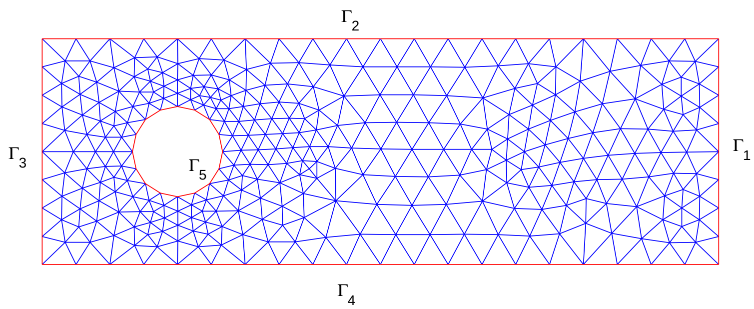


Figure 1: Geometry and mesh for the cylinder problem.

As boundary conditions, we take $u = 10$ on the cylinder (Γ_5), $\partial_n u = 0$ on Γ_1 and $u = 0$ on the rest of the boundary.

Check your answer: Plot your solution using the `pdesurf` command and compare your solution to Figure 2.

Problem 3

Preparation

Go to the directory `problem3`. Create a file called `ConvDiffSolver.m` and write your program in this file. You also have to write a new file called `ConvDiff.m`, where you specify the the variational formulation. Use the file `Poisson.m` as a template.

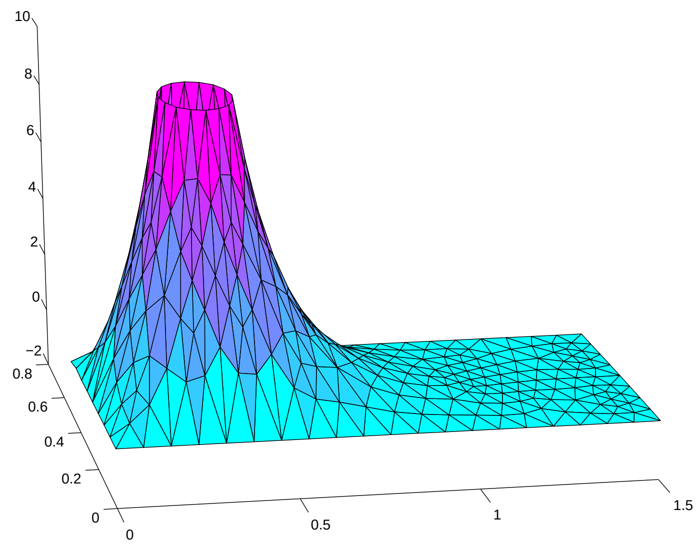


Figure 2: A plot of the solution for Problem 2.

Problem

Now solve the convection–diffusion equation, hej

$$
\begin{aligned}
b \cdot \nabla u - \nabla \cdot (a \nabla u) &= f \quad \text{in } \Omega, \\
-a \partial_n u &= \gamma(u - g_D) + g_N \quad \text{on } \Gamma,
\end{aligned}
\tag{5}
$$

with the same boundary conditions as in the previous problem, $a = 1$, and the convection given by $b = (20, 0)$, corresponding to a flow from left to the right in the domain.

Check your answer: Plot your solution using the `pdesurf` command and compare your solution to Figure 3.

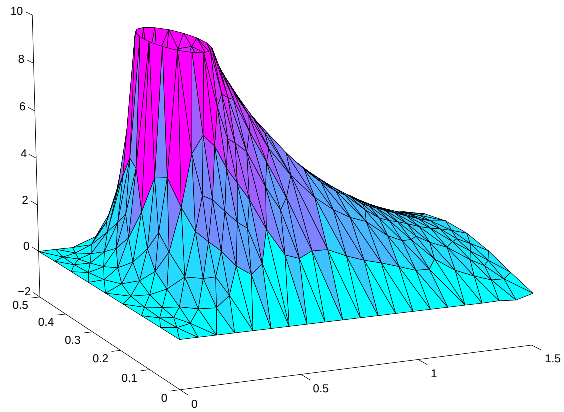


Figure 3: A plot of the solution for Problem 3.

Hints

Problem 1

The variational form of (1) that should be specified in the file `Poisson.m` is

```
du'*dv*dx + g(x,d,t)*u*v*ds
```

for the left-hand side and

```
f(x,d,t)*v*dx + (g(x,d,t)*gd(x,d) - gn(x,d,t))*v*ds
```

for the right-hand side. To assemble the matrix, write

```
A = AssembleMatrix(p, e, t, 'Poisson', [], 0)
```

and to assemble the vector, write

```
b = AssembleVector(p, e, t, 'Poisson', [], 0)
```

The last two arguments are not used in this computer session, but will be very important in the coming computer sessions.

Problem 2

Use the argument `d` in the functions `g()`, `gd()`, and `gn()` to specify the boundary conditions for the different boundaries. Write an `if`-statement or a `switch` to specify different values of γ, g_D, g_N.

Problem 3

The extra term that you need to add in the variational formulation has the form

```
[20 0] * du * v
```

Solutions

Make sure that you really try to solve each problem before looking at the solutions. Have you really tried to solve the problem or should you try again before looking at the solution?

The solutions are available on the web page of this session under *Solutions to problems.*

About

This Computer Session is part of the Body and Soul educational program. More information can be found at

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http://www.phi.chalmers.se/bodysoul/
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This Computer session is written by Anders Logg.